

Simple Cohen-Macaulay Codimension 2 Singularities

Anne Frühbis-Krüger	Alexander Neumer
Inst. f. Alg. Geom.	Stiftung IHF
Leibniz Univ. Hannover	Bremser Str. 79
30167 Hannover	67063 Ludwigshafen/Rh.
Germany	Germany

September 4, 2008

Abstract

In this article, we provide a complete list of simple isolated Cohen-Macaulay codimension 2 singularities together with a list of adjacencies which is complete in the case of fat point and space curve singularities.

1 Introduction

In this article, we determine a complete classification of simple Cohen-Macaulay codimension 2 singularities and in the case of fat points and curves their complete adjacency list. Classification is done up to isomorphism of germs; a singularity is called simple, if it can only deform into finitely many different isomorphism classes – in other words, if the modality of the singularity is zero. Cohen-Macaulay codimension 2 singularities are particularly important, as not all of them are complete intersections, but they are, nevertheless, unobstructed and the theorem of Hilbert-Burch provides a powerful tool for describing these singularities and their deformations.

Arnold was the first to choose modality as the criterion in his pioneering work [Arn]. There he stated the famous ADE-list of simple hypersurface singularities. About a decade later Giusti [Giu] gave a list of simple complete intersection singularities; shortly thereafter Wall [Wal] extended this to a classification of unimodal isolated complete intersection singularities. But there are singularities of modality 0 which are not complete intersections, as the following two examples show: the three coordinate axes in $(\mathbb{C}^3, 0)$, given by xy, xz, yz , can be checked to be simple by direct computation, and the singularity in $(\mathbb{C}^6, 0)$ given by the 2-minors of the 3×2 matrix, whose entries are exactly the variables, does not permit any non-trivial deformations ($T^1 = 0$). In the case of space curve

singularities, Giusti's list was completed by the list of simple non-complete-intersection space curves by Frühbis-Krüger [FK1] another decade later. For readers' convenience, the above-mentioned lists of simple singularities are included in the appendix, as we are computing adjacencies to them.

For the first task in this article, the classification, the main tools are the Hilbert-Burch theorem ([Bur]) and its generalization to deformations of singularities (cf. [Sch], [Art]). These allow us to describe the singularity by the presentation matrix and the flat deformations by perturbations of this matrix. So the role, which is played by the ideal in the case of complete intersection singularities, is now taken over by the presentation matrix of this ideal; this leads to reformulations of the T^1 and a finite determinacy statement in terms of the presentation matrix as is shown in detail in [FK1].

In the proof of simplicity in each of the above-mentioned classifications by Giusti and Frühbis-Krüger, also a large number of adjacency relations has been determined, but it was often unnecessary to consider all adjacency relations; so the adjacency lists stated there cannot be assumed to be complete. For instance, the adjacency list for the simple space curves from Giusti's list was not completed before the late 1990s (cf. [S-V]), although it had already been known to be incomplete for nearly a decade (cf. [Gor]), and even then the adjacencies to plane curve singularities were not determined.

The main tool for excluding adjacencies is semicontinuity of numerical invariants, such as the Tjurina number, Milnor number and Delta invariant. But even in the case of space curve singularities, this tool is far from being powerful enough to decide all possible cases. In this situation, determining the complete list of adjacencies would not have been possible without systematic use of the computer algebra system SINGULAR ([Sin]) and, in particular, the partial standard bases algorithm, a specialized tool for computing simultaneously in families of singularities (cf. [FK2], [FK3]). For each of the singularities, this allowed us to compute the stratum in the base of the versal family, where the Tjurina number is exactly one less than the one of the original singularity. Equipped with the additional knowledge, which singularities of this particular value of τ were appearing in the versal family, it was then possible to reduce the cases, which had to be excluded by explicit calculation, to just 3. In higher dimensions, the same method can be applied to those singularities which are not part of a series, but this is not part of this article.

With these lists, we hope to provide a set of examples of singularities together with their adjacencies also in cases which are not hypersurfaces, curves or surfaces. We should like to thank Gerhard Pfister and the whole algebraic geometry group at the University of Kaiserslautern as well as the developers of the computer algebra system SINGULAR [Sin] for many fruitful discussions. We would also like to thank Jan Stevens for pointing out an omission in an earlier version of this article and for several helpful remarks.

2 Basics

For a detailed discussion of the methods used to study germs of Cohen-Macaulay codimension 2 singularities with respect to contact-equivalence, see [FK1]. In short, we can say that using the Hilbert-Burch theorem, all Cohen-Macaulay germs of codimension 2 can be expressed as the maximal minors of $(n+1) \times n$ -matrices M and vice versa. In the same way, flat deformations can be represented by perturbations of the matrix M and any perturbation gives rise to a flat deformation (cf. [Bur], [Sch]).

Classification up to contact-equivalence means that two singularities are considered equivalent, if the germs are isomorphic.¹ The action of the contact-group translates directly to the application of coordinate changes and row and column operations on M . A singularity is called simple, if it can only deform into finitely many different equivalence classes (types) of singularities.

Definition 2.1 ([FK1]) *Let M be a $(n+1) \times n$ matrix with entries in $\mathbb{C}\{x_1, \dots, x_m\}$. M is called quasihomogeneous of type $(D; a) \in \text{Mat}((n+1) \times n; \mathbb{N}) \times \mathbb{N}^m$, if*
a) all entries M_{ij} are quasihomogeneous of degree D_{ij} with respect to the weight vector a
b) there are relative row and column weights, i.e.
 $D_{ij} - D_{ik} = D_{lj} - D_{lk}$ *for all $1 \leq i, l \leq n+1, 1 \leq j, k \leq n$*
Let N be another $(n+1) \times n$ matrix with entries in $\mathbb{C}\{x_1, \dots, x_m\}$. The relative matrix weight of N with respect to $(D; a)$ is given by

$$v_{(D; a)}(N) := \inf_{j, i} \{v_a(N_{ij}) - D_{ij}\}$$

Lemma 2.2 ([FK1]) *Let $(X, 0)$ be an isolated Cohen-Macaulay codimension 2 singularity which is quasihomogeneous w.r.t. some weight vector a . Then it is possible to find a presentation matrix M (describing the singularity X) which is quasihomogeneous of type $(D; a)$ for a suitable $D \in \text{Mat}((n+1) \times n; \mathbb{N})$.*

For a consistent notation in the discussion, it is also necessary to reformulate $T_{X,0}^1$ and the finite determinacy criterion in terms of the presentation matrix²:

Lemma 2.3 ([FK1]) $T_{X,0}^1$ *is given by*

$$T_{X,0}^1 \cong \text{Mat}(n+1, n; \mathbb{C}\{x_1, \dots, x_n\}) / (J(M) + \text{Im}(g))$$

where $J(M)$ is the submodule generated by the matrices of the form

$$\begin{pmatrix} \frac{\partial M_{11}}{\partial x_j} & \dots & \frac{\partial M_{1n}}{\partial x_j} \\ \vdots & & \vdots \\ \frac{\partial M_{(n+1)1}}{\partial x_j} & \dots & \frac{\partial M_{(n+1)n}}{\partial x_j} \end{pmatrix} \quad \forall 1 \leq j \leq m$$

¹We use the symbol \sim_C to indicate contact-equivalence.

²If we are considering a singularity $X, 0$ in the notation of its presentation matrix M , we often also denote $T_{X,0}^1$ by $T^1(M)$

and g is the map

$$\begin{aligned} \text{Mat}(n+1, n+1; \mathbb{C}\{x_1, \dots, x_m\}) \oplus \text{Mat}(n, n; \mathbb{C}\{x_1, \dots, x_m\}) \\ \xrightarrow{g} \text{Mat}(n+1, n; \mathbb{C}\{x_1, \dots, x_m\}) \end{aligned}$$

mapping $(A, B) \mapsto AM + MB$.

By using the relative matrix weight, $T_{X,0}^1$ can be regarded as a graded module $\bigoplus_{v \in \mathbb{Z}} T_v^1(M)$.

This can in turn be used to formulate an explicit determinancy criterion for isolated quasihomogeneous Cohen-Macaulay codimension 2 singularities:

Lemma 2.4 ([FK1]) *Let M be a $(n+1) \times n$ matrix with entries in the maximal ideal of $\mathbb{C}\{x_1, \dots, x_m\}$, quasihomogeneous of type $(D; a)$ and defining an isolated singularity. Let N be another $(n+1) \times n$ matrix with entries in $\mathbb{C}\{x_1, \dots, x_m\}$, such that*

$$v_{(D;a)}(N) > \beta = \sup\{0, \alpha\}$$

where $\alpha = \sup\{\nu \in \mathbb{Z} | T_\nu^1(M) \neq 0\}$.

Then $M + N \sim_C M$.

3 Candidates in dimension ≥ 4

3.1 Reduction of the problem

Let G_0 be the \mathbb{C} -vector space of all quasihomogeneous $(n+1) \times n$ matrices of type (D, a) for an arbitrary fixed positive integer n . For a generic matrix $M \in G_0$, the kernel of the natural surjection $G_0 \longrightarrow T_0^1(M)$ is generated by the set $S_1 \cup S_2 \cup S_3$ where

$$\begin{aligned} S_1 &= \left\{ b_{i,a_j} \frac{\partial M}{\partial x_j} \mid 1 \leq i \leq r(a, a_j), 1 \leq j \leq m \right\} \\ S_2 &= \left\{ b_{i,D_{l1}-D_{j1}} Z_{jl} \mid 1 \leq i \leq r(a, D_{l1} - D_{j1}), 1 \leq j, l \leq n+1 \right\} \\ S_3 &= \left\{ b_{i,D_{1l}-D_{1j}} S_{jl} \mid 1 \leq i \leq r(a, D_{1l} - D_{1j}), 1 \leq j, l \leq n \right\}. \end{aligned}$$

Here $\{b_{1,d}, \dots, b_{r(a,d),d}\}$ is the set of monomials of weighted degree d , $r(a, d)$ its cardinality; Z_{jl} denotes the $(n+1) \times n$ -matrix, having the l -th row of M as its j -th row and all other entries 0. In the same way, S_{jl} is the matrix with the l -th column of M as its j -th column.

Recall that a singularity is called simple, if it can only deform into finitely many equivalence classes of singularities. Thus counting degrees of freedom shows that a singularity defined by an element $M \in G_0$ of type $(D; a)$ can only be simple if the dimension of G_0 , which is just $\sum_{ij} r(a, D_{ij})$, does not exceed the dimension of the above kernel, i.e. the number s of linearly independent

elements of S_1, S_2 and S_3 . Since we always have two relations, the Euler relation and $\sum S_{ii} = \sum Z_{jj}$, this means that the inequality $\#S_1 + \#S_2 + \#S_3 - 2 \geq \sum_{ij} r(a, D_{ij})$ has to hold.

Lemma 3.1 *A $(n+1) \times n$ -matrix M can only define a simple isolated codimension 2 singularity in $(\mathbb{C}^m, 0)$ if $n < 3$ and $m < 7$.*

Proof: We will use a modified version of the above counting argument to prove that the number of different variables occuring in the 1-jet of M has to be greater than $n^2 + n - 2$ whereas the total number of variables may not exceed $n^2 + n$.

Step 1 (An upper bound for the number of variables) Let us first suppose that the total number of variables m exceeds $n^2 + n$. Consider a generic $(n+1) \times n$ -matrix, quasihomogeneous w.r.t. the weights $a = (1, \dots, 1)$ and the degrees $D_{ij} = 1$. The matrix contains $n^2 + n$ different linear entries and is hence equivalent to

$$\begin{pmatrix} x_1 & \dots & x_n \\ \vdots & & \vdots \\ x_{n^2+1} & \dots & x_{n(n+1)} \end{pmatrix}.$$

This matrix does not allow any non-trivial perturbations and any other matrix of the corresponding size in m variables is adjacent to it; on the other hand, a direct computation shows that it defines a non-isolated singularity. Therefore isolated singularities can only occur for $m \leq n^2 + n$.

Step 2 (A counting argument on the 1-jet) Let N be a generic $(n+1) \times n$ -matrix, quasihomogeneous w.r.t. the weights $a = (1, \dots, 1)$ and the degrees $D_{ij} = 1$. For these weights, the kernel of the map from G_0 to $T_0^1(N)$ in the above argument is generated by $\#S_1 + \#S_2 + \#S_3 - 2 = m^2 + (n+1)^2 + n^2 - 2$ elements. Comparing this to $m(n^2 + n)$, the dimension of $T_0^1(N)$, we get the inequality $m^2 + (n+1)^2 + n^2 - 2 \geq m(n^2 + n)$ as a necessary condition for N being simple. This simplifies to $m^2 > (m-2)(n^2 + n)$. First of all, we see directly from this inequality that for $n > 2$, m has to be at least 10. Using this additional information, we can now simplify our condition to

$$\begin{aligned} m+3 > m+2 + \frac{4}{m-2} &= \frac{(m+2)(m-2)+4}{m-2} = \\ &= \frac{m^2}{m-2} > \frac{(m-2)(n^2+n)}{m-2} = n^2 + n. \end{aligned}$$

Since N was a generic matrix of this type, the 1-jet of any matrix M of appropriate size with entries in the maximal ideal of $\mathbb{C}\{x_1, \dots, x_m\}$ is adjacent to N .

Combining this with the result of step 1, we see that $n^2 + n - 2 \leq m \leq n^2 + n$, but we still do not have a bound for the number of variables actually appearing in the 1-jet nor do we have a bound for the size of the matrix. To obtain these two, we now pass to another set of weights: Let N be a generic $(n+1) \times n$ -matrix,

quasihomogeneous w.r.t. the weights³

$$a = (\underbrace{2, \dots, 2}_p, \underbrace{1, \dots, 1}_{m-p}) \quad D_{ij} = 2.$$

For these weights, we obtain

$$\#S_1 + \#S_2 + \#S_3 - 2 = n^2 + (n+1)^2 + p(p + (m-p) \cdot \frac{m-p+1}{2}) + (m-p)^2$$

and

$$\dim_{\mathbb{C}} T_0^1(N) = n(n+1)(p + (m-p) \cdot \frac{m-p+1}{2}).$$

Plugging in each of the three possible values of m , we obtain the following table whose entries are the difference of the number of degrees of freedom and the number of possible entries minus 2. Hence simple singularities can only occur if the entry has a positive value.

	$m = n^2 + n - 2$	$m = n^2 + n - 1$	$m = n^2 + n$
$p = n^2 + n$	-	-	$2n^2 + 2n - 1$
$p = n^2 + n - 1$	-	$n^2 + n$	$n^2 + n$
$p = n^2 + n - 2$	3	2	1
$p = n^2 + n - 3$	$-n^2 - n + 6$	$-n^2 - n + 3$	$-n^2 - n - 1$
$p = n^2 + n - 3$	$-2n^2 - 2n + 7$	-	-

We immediately see that at least $n^2 + n - 2$ variables have to appear in the 1-jet resp. $n^2 + n - 3$ in the case $n = 2$, $m = 4$.

Step 3 (Excluding non-isolated singularities) The conditions obtained in steps 1 and 2 imply that (for $n > 2$) $j_1 M$ has to contain at least $n^2 + n - 2$ different variables. But then $j_1 M$ is contact-equivalent to a matrix of the form

$$j_1 M \sim_C \begin{pmatrix} x_{11} & \dots & x_{(n-1)1} & x_{n1} & x_{(n+1)1} \\ \vdots & & & & \vdots \\ x_{1(n-1)} & \dots & x_{(n-1)(n-1)} & \alpha & x_{(n+1)(n-1)} \\ x_{1n} & \dots & x_{(n-1)n} & x_{nn} & \beta \end{pmatrix}$$

with $\alpha, \beta \in \mathfrak{m}$, and by another coordinate change we see that M is of the same form.

Direct computation shows that the ideal of the singular locus of M is contained in $\langle x_{11}, \dots, x_{1n}, x_{21}, \dots, x_{2n} \rangle$. This component is obviously not zero-dimensional, and hence the singularity defined by M cannot be isolated.

□

³Any matrix whose 1-jet only involves p of the m variables is adjacent to this matrix and hence this is the set of weights to consider for determining the least number of variables appearing in the 1-jet of a simple singularity of given size $n \times (n+1)$ and number of variables m .

We can now restrict our consideration to the case of $(n + 1) \times n$ -matrices with $n \leq 2$. In the case $n = 1$, the singularity is a complete intersection; a complete classification of simple isolated singularities in this situation can be found in [Giu].

In the remaining case of 3×2 -matrices with $n = 2$, the calculations of step 2 of the preceding proof imply that simple isolated singularities can only occur for:

dimension m	possible 1-jet candidates
3	3 variables
4	3, 4 variables
5	4, 5 variables
6	4, 5, 6 variables

Remark: Unfortunately, this method gives no bound for the case of fat points in $(\mathbb{C}^2, 0)$. Thus we will study fat points separately in section 4.

As the next step, we will classify the possible candidates of 1-jets with 4 or more variables:

3.2 1-jet candidates

First we will classify the possible candidates of 1-jets with 4 or more variables:

Lemma 3.2 *Let M be a 3×2 -matrix with entries in the maximal ideal of $\mathbb{C}\{x_1, \dots, x_m\}$. Then $j_1 M$ is contact-equivalent to one of the jets in the following tables*

6 variables	
$J^{(6,1)}$	$\begin{pmatrix} x & y & v \\ z & w & u \end{pmatrix}$

5 variables	
$J^{(5,1)}$	$\begin{pmatrix} x & y & v \\ z & w & x \end{pmatrix}$
$J^{(5,2)}$	$\begin{pmatrix} x & y & v \\ z & w & 0 \end{pmatrix}$

4 variables	
$J^{(4,1)}$	$\begin{pmatrix} w & y & x \\ z & w & y \end{pmatrix}$
$J^{(4,2)}$	$\begin{pmatrix} w & y & x \\ z & w & 0 \end{pmatrix}$
$J^{(4,3)}$	$\begin{pmatrix} 0 & y & x \\ z & w & 0 \end{pmatrix}$
$J^{(4,4)}$	$\begin{pmatrix} x & y & z \\ z & w & 0 \end{pmatrix}$
$J^{(4,5)}$	$\begin{pmatrix} x & y & 0 \\ z & w & 0 \end{pmatrix}$
$J^{(4,6)}$	$\begin{pmatrix} x & y & z \\ w & 0 & 0 \end{pmatrix}$

or is contact-equivalent to a 1-jet containing only 3 or less variables.

To simplify notation, we will abbreviate the case k in dimension d as $J^{(d,k)}$.

Proof: Because the arguments for all these cases work in a similar way, we will concentrate on the presentation of the case where j_1M contains exactly 5 variables.

By applying several coordinate changes and row and column operations we may assume w.l.o.g. that

$$j_1M \sim_C \begin{pmatrix} x & y & v \\ z & w & \alpha \end{pmatrix} \quad \text{with } \alpha \in \mathfrak{m}.$$

If $\alpha = 0$, we have a 1-jet of type $J^{(5,2)}$. Otherwise we can write $\alpha = \alpha_1x + \alpha_2y + \alpha_3z + \alpha_4w + \alpha_5v$ with $\alpha_i \in \mathbb{C}$. By contact-equivalence, we get

$$j_1M \sim_C \begin{pmatrix} x & y & v \\ z & w & \alpha'_1x + \alpha'_2y \end{pmatrix} \quad \text{with } \alpha'_1, \alpha'_2 \in \mathbb{C}.$$

We may now assume $\alpha'_1 \neq 0$ (by exchanging the first and second column and exchanging the roles of x and y respectively of z and w) and obtain

$$j_1M \sim_C \begin{pmatrix} x & y & v \\ z & w & x \end{pmatrix}.$$

□

The possible 1-jets which contain exactly three variables were already determined in [FK1]:

<i>3 variables</i>			
$J^{(3,1)}$	$\begin{pmatrix} z & y & x \\ 0 & x & y \end{pmatrix}$	$J^{(3,2)}$	$\begin{pmatrix} z & y & 0 \\ 0 & x & y \end{pmatrix}$
$J^{(3,3)}$	$\begin{pmatrix} z & y & 0 \\ y & x & z \end{pmatrix}$	$J^{(3,4)}$	$\begin{pmatrix} z & 0 & 0 \\ 0 & x & y \end{pmatrix}$
$J^{(3,5)}$	$\begin{pmatrix} z & y & x \\ y & 0 & 0 \end{pmatrix}$	$J^{(3,6)}$	$\begin{pmatrix} z & 0 & x \\ 0 & z & y \end{pmatrix}$

The preceeding two lemmata provide a classification of all 1-jets that may occur in simple isolated Cohen-Macaulay codimension 2 singularities in $(\mathbb{C}^p, 0)$, $p \geq 4$. Now we will check whether these 1-jets lead to simple singularities. Since we already know that we will not get simple singularities for all 1-jets in every dimension, we will start by regarding the candidates in the smallest dimension.

3.3 Singularities in $(\mathbb{C}^4, 0)$

Because we have seen that 1-jets containing only 2 or fewer variables cannot be simple in dimension 4, we have to consider only the 1-jets with 3 and 4 variables.

Theorem 3.3 *The following table shows the list of simple isolated Cohen-Macaulay codimension 2 singularities in $(\mathbb{C}^4, 0)$.*

Jet-Type	Type	Presentation Matrix		τ	Name of Triple Point in [Tju]
$J^{(4,1)}$	$\Lambda_{1,1}$	$\begin{pmatrix} w & y & x \\ z & w & y \end{pmatrix}$		2	$A_{0,0,0}$
$J^{(4,2)}$	$\Lambda_{k,1}$	$\begin{pmatrix} w & y & x \\ z & w & y^k \end{pmatrix}$	$k \geq 2$	$k + 1$	$A_{0,0,k-1}$
$J^{(4,3)}$	$\Lambda_{k,l}$	$\begin{pmatrix} w^l & y & x \\ z & w & y^k \end{pmatrix}$	$k \geq l \geq 2$	$k + l$	$A_{0,l-1,k-1}$
$J^{(4,4)}$		$\begin{pmatrix} z & y & x \\ x & w & y^2 + z^k \end{pmatrix}$	$k \geq 2$	$k + 3$	$C_{k+1,0}$
		$\begin{pmatrix} z & y & x \\ x & w & yz + y^k w \end{pmatrix}$	$k \geq 1$	$2k + 4$	$B_{2k+2,0}$
		$\begin{pmatrix} z & y & x \\ x & w & yz + y^k \end{pmatrix}$	$k \geq 3$	$2k + 1$	$B_{2k-1,0}$
		$\begin{pmatrix} z & y & x \\ x & w & z^2 + yw \end{pmatrix}$		7	D_0
		$\begin{pmatrix} z & y & x \\ x & w & z^2 + y^3 \end{pmatrix}$		8	F_0
$J^{(3,1)}$		$\begin{pmatrix} z & y + w^l & w^m \\ w^k & y & x \end{pmatrix}$	$k, l, m \geq 2$	$k + l + m - 1$	$A_{k-1,l-1,m-1}$
$J^{(3,2)}$		$\begin{pmatrix} z & y & x^l + w^2 \\ w^k & x & y \end{pmatrix}$	$k, l \geq 2$	$k + l + 2$	$C_{l+1,k-1}$
		$\begin{pmatrix} z & y + w^l & xw \\ w^k & x & y \end{pmatrix}$	$k, l \geq 2$	$k + 2l + 1$	$B_{2l,k-1}$
		$\begin{pmatrix} z & y & xw + w^l \\ w^k & x & y \end{pmatrix}$	$k \geq 2, l \geq 3$	$k + 2l$	$B_{2l+1,k-1}$
		$\begin{pmatrix} z & y + w^2 & x^2 \\ w^k & x & y \end{pmatrix}$	$k \geq 2$	$k + 6$	D_{k-1}
		$\begin{pmatrix} z & y & x^2 + w^3 \\ w^k & x & y \end{pmatrix}$	$k \geq 2$	$k + 7$	F_{k-1}
$J^{(3,3)}$		$\begin{pmatrix} z & y & xw + w^k \\ y & x & z \end{pmatrix}$		$3k + 1$	H_{3k}
		$\begin{pmatrix} z & y & xw \\ y & x & z + w^k \end{pmatrix}$		$3k + 2$	H_{3k+1}
		$\begin{pmatrix} z & y & xw \\ y + w^k & x & z \end{pmatrix}$		$3k + 3$	H_{3k+2}
		$\begin{pmatrix} z & y & w^2 \\ y & x & z + x^2 \end{pmatrix}$		8	
		$\begin{pmatrix} z & y & x^3 + w^2 \\ y & x & z \end{pmatrix}$		9	

		$\begin{pmatrix} z & y & x^2 \\ y & x & z + w^2 \end{pmatrix}$		9	
--	--	--	--	---	--

Proof: We will consider each of the possible 1-jets from lemma 3.2 separately. As the proof that a singularity cannot be simple by variants of the counting argument always has the same structure, we only list the non-simple singularities and the respective weights here:

Jet-Type	Presentation Matrix	\underline{a}	D	τ
$J^{(4,4)}$	$\begin{pmatrix} z & y & x \\ x & w & z^2 + y^4 \end{pmatrix}$	$\begin{pmatrix} 2 & 1 & 3 \\ 3 & 2 & 4 \end{pmatrix}$	$(3 \ 1 \ 2 \ 2)$	11
	$\begin{pmatrix} z & y & x \\ x & w & y^3 + z^3 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}$	$(2 \ 1 \ 1 \ 2)$	10
$J^{(4,5)}$	$\begin{pmatrix} z & y & \alpha \\ x & w & \beta \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}$	$(1 \ 1 \ 1 \ 1)$	13
$J^{(4,6)}$	$\begin{pmatrix} x & y & z \\ w & \alpha & \beta \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}$	$(1 \ 1 \ 1 \ 2)$	9
$J^{(3,2)}$	$\begin{pmatrix} z & y & \alpha \\ w^2 & x & y + w^2 \end{pmatrix}$	$\begin{pmatrix} 3 & 2 & 3 \\ 2 & 1 & 2 \end{pmatrix}$	$(1 \ 2 \ 3 \ 1)$	11
	$\begin{pmatrix} z & y + w^3 & x^2 + \alpha w^4 \\ w^2 & x & y \end{pmatrix}$	$\begin{pmatrix} 3 & 3 & 4 \\ 2 & 2 & 3 \end{pmatrix}$	$(2 \ 3 \ 3 \ 1)$	12
$J^{(3,4)}$	$\begin{pmatrix} y & z & \alpha \\ \beta & y + \gamma & z + \delta \end{pmatrix}$	$\begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}$	$(1 \ 2 \ 2 \ 1)$	15
$J^{(3,6)}$	$\begin{pmatrix} z & y & x \\ \alpha & \beta & y + \gamma \end{pmatrix}$	$\begin{pmatrix} 4 & 6 & 4 \\ 6 & 8 & 6 \end{pmatrix}$	$(4 \ 6 \ 4 \ 3)$	15

$J^{(4,1)}$: Let

$$M = \begin{pmatrix} x & y & z \\ z & w & y \end{pmatrix}.$$

By direct computation, $T^1(M)$ is generated by

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus any deformation of M is of the form

$$M'(\alpha, \beta) = \begin{pmatrix} x & y & z + \alpha \\ z & w & y + \beta \end{pmatrix}$$

with $\alpha, \beta \in \mathbb{C}$. By direct computation, the singular locus of $M'(\alpha, \beta)$ is

$$\begin{aligned} \text{Sing}(M'(\alpha, \beta)) = \langle & \beta^2, \alpha\beta, \alpha^2, w\beta, 5w\alpha - \beta^2, w^2, 5z\beta + 4\alpha\beta, \\ & 5z\alpha + 3\alpha^2, zw + w\alpha, 2z^2 + 3z\alpha + \alpha^2, 2y\beta - zw + \beta^2, 3y\alpha + 2z\beta + 4\alpha\beta, \\ & yw + w\beta, yz + y\alpha + z\beta + \alpha\beta, 2y^2 + 3y\beta + \beta^2, x\beta - 5z^2 - 3z\alpha, \\ & x\alpha, xw - 4y - 2y\alpha - 2z\beta - \alpha\beta, xz + x\alpha, xy + 2z^2 + z\alpha, x^2 \rangle \end{aligned}$$

Because this ideal contains α^2 and β^2 , $M'(\alpha, \beta)$ defines a smooth surface if either α or $\beta \neq 0$. Hence M defines a simple singularity we will call $\Lambda_{1,1}$. Since M is 1-determined, every matrix of type $J^{(4,1)}$ is contact-equivalent to M .

$J^{(4,2)}$: In this case M is contact-equivalent to a matrix

$$\begin{pmatrix} w & y & x \\ z & w + \alpha & \beta \end{pmatrix}$$

with $\alpha, \beta \in \mathfrak{m}^2$. Moreover, we can get rid of α and all terms of β involving x, z, w by row and column operations and by appropriate coordinate changes of w . This leads to the matrix

$$\begin{pmatrix} w & y & x \\ z & w & y^k \cdot (1 + \gamma(y)) \end{pmatrix}$$

where $\gamma \in \mathfrak{m}$. Dividing the last column by the unit $(1 + \gamma(y))$ and performing an appropriate coordinate change in x , we then obtain the desired structure of the matrix.

For the proof of simplicity, we can proceed in the same way as in the previous case and obtain that only adjacencies to singularities of the same series with lower τ , of the type $J^{(4,1)}$ and of the A -series (i.e. $(w, y^k + xz)$) can appear.

$J^{(4,3)}$: This case is strictly analogous to the case $J^{(4,2)}$, with the only difference that we obtain at most adjacencies to the singularities of the same series with lower τ , to the series $J^{(4,2)}$, to the singularity $J^{(4,1)}$ and to the A -series.

$J^{(4,4)}$: A matrix with 1-jet of type $J^{(4,4)}$ is of the structure

$$\begin{pmatrix} z & y & x + \alpha \\ x & w & \beta \end{pmatrix}$$

where $\alpha, \beta \in \mathfrak{m}^2$. All terms of α can be cancelled in the same way as in the case $J^{(4,2)}$. Regarding β only terms in y and z and terms of the form $y^k w$ cannot be killed. By the table of non-simple singularities, we can conclude that β cannot be of order 3 or higher. A direct calculation then shows that the 2-jet of β has to be one of the following 7: $y^2 + z^2, yz + yw,$

$z^2 + yw, y^2, yz, z^2$ and yw . In the first three cases the corresponding 2-jet of the matrix is already 2-determined which implies that each gives rise to exactly one singularity. In the fourth case, the only monomial of higher degree which may occur is a power of z , leading to the first series in the list. In the 5th case, pure powers of y and terms $y^k w$ are the only terms that cannot be cancelled, but for determinacy reasons more than one of them cannot occur simultaneously; this gives rise to the second series. In the last two cases, the only possibility which is not excluded by the list of non-simple singularities (case $J^{(4,4)}$, lines 1 and 2) is $z^2 + y^3$.

For the proof of simplicity, we need to study two different questions here: first of all, we have to find out whether some singularities from the series are adjacent to non-simple ones and secondly, we have to find out whether any of these singularities can deform into a non-simple one of a different 1-jet. The first question can be answered quite easily by observing that whenever a term y^2 is present, the singularity is in the first series, and whenever a term yz , but no y^2 is present, the term z^2 may be killed by a coordinate transform in y and a subsequent column operation on the second column.

The second question involves a simple, but rather lengthy explicit calculation which shows that a singularity of type $J^{(4,4)}$ can only deform into singularities of types $A_k, D_k, J^{(4,1)}, J^{(4,2)}$ and $J^{(4,3)}$: More precisely, we consider the versal family, deduce conditions deciding when a point in the base space allows a singularity and then determine the occurring kinds of singularities. The versal family in our case is

$$\begin{pmatrix} z & y & x + \alpha \\ x & w & \beta + \gamma y + \delta w + \varepsilon z + p(y, z, w) \end{pmatrix},$$

$\alpha, \dots, \varepsilon \in \mathbb{C}$ and $p \in \mathfrak{m}^2$. As the whole calculation is rather lengthy we only sketch a part of it, namely the case $\alpha = 0$: For fixed $\beta, \dots, \varepsilon$, singularities may only occur at points where the order of the lower right hand entry of the matrix is at least 1. By the structure of the matrix, we see that for $\beta \neq 0$ at most a D_k singularity ($x - zw, zw^2 - \dots$) may occur, if y is non-zero at this point, at most an A_k singularity ($w - yx, x^2 + y^2 + \dots$ or $w - yx, x^2 - yz$) if the z coordinate is non-zero, and no singularities at other points. If $\beta = 0$ and the 1-jet is not of type $J^{(4,4)}$, then $\gamma \neq 0$ implies that we are dealing with a singularity of type $J^{(4,1)}$, $\gamma = 0, \delta \neq 0$ leads to $J^{(4,2)}$ and $\gamma = \delta = 0, \varepsilon \neq 0$ leads to $J^{(4,3)}$.

$J^{(4,5)}$: As the generic matrix of this jet appears among the non-simple singularities, we cannot get any candidates here.

$J^{(4,6)}$: No simple singularities possible, same argument as for $J^{(4,5)}$.

$J^{(3,1)}$: In this case, the matrix is contact-equivalent to

$$\begin{pmatrix} z & y + \alpha & x + \beta \\ \gamma & x & y \end{pmatrix}$$

where $\alpha, \beta, \gamma \in \mathfrak{m}^2$; moreover, we can achieve that α, β and γ only contain terms in w by appropriate row and column operations and coordinate changes in x, y and z . Therefore the matrix can be written as

$$\begin{pmatrix} z & y + a_1 w^{k_1} & x + a_2 w^{k_2} \\ a_3 w^{k_3} & x & y \end{pmatrix}$$

where $a_1, a_2, a_3 \in \mathbb{C}\{w\}$ and either $a_i = 0$ or $a_i(0) \neq 0$ for $1 \leq i \leq 3$. Here it is most convenient to pass to an equivalent way of writing the one jet (by a coordinate change $x \mapsto \frac{1}{2}(x+y)$ and $y \mapsto \frac{1}{2}(x-y)$ followed by appropriate row and column operations), in which the matrix can then be stated as

$$\begin{pmatrix} z & y + b_1 w^{k_1} & b_2 w^{k_2} \\ b_3 w^{k_3} & y & x \end{pmatrix}$$

As the matrix would not describe a finitely determined singularity if any of the three b_i were zero, we easily achieve that $b_2 = b_3 = 1$ and with a little more work that also $b_1 = 1$. Writing the matrix in this way, it is obviously quasihomogeneous and we hence write this normal form in the list. On the other hand, transforming it back into the other form facilitates comparisons in subsequent adjacency calculations:

$$\begin{pmatrix} z & y + w^l + w^m & x + w^l - w^m \\ w^k & x & y \end{pmatrix}$$

To prove simplicity, we proceed in the same way as outlined in $J^{(4,4)}$ and obtain that the singularity can at most be adjacent to the following singularities/series of singularities: A -series, $J^{(4,1)}$, $J^{(4,2)}$ and $J^{(4,3)}$. Since all members of these series are simple as we already proved, these singularities are simple, as well.

$J^{(3,2)}$: By the same argument as in the case $J^{(3,1)}$, the matrix has to be of the structure

$$\begin{pmatrix} z & y + \alpha & \beta \\ w^k & x & y \end{pmatrix}$$

where $\alpha, \beta \in \mathfrak{m}^2$, α only involving terms in w , β only involving terms in x and w . A matrix of this structure does not define an isolated singularity if neither α nor β contain a pure power in w . Moreover, if the order of β is at least 3, the singularity cannot be simple due to an adjacency to the non-simple singularities ($J^{(3,2)}$ line 1). Let us start with the case that β contains the term w^2 , which may w.l.o.g. be written as:

$$\begin{pmatrix} z & y + \alpha & a_1 x^{k_1} + a_2 x^{k_2} w + w^2 \\ w^k & x & y \end{pmatrix}$$

where $a_1, a_2 \in \mathbb{C}\{x, w\}$ and either $a_i = 0$ or $a_i(0) \neq 0$ for each of them. By an appropriate coordinate change in w followed by subtracting an appropriate multiple of the 2nd column from the first one and another

coordinate change in z , we can get rid of the term $a_2x^{k_2}w$ changing of course the term $a_1x^{k_1}$ to some $\tilde{a}_1x^{k_1}$. By determinacy, the terms of α may be omitted which leads to the first series.

If, on the other hand, the term w^2 is not present in β , we are dealing with a matrix of the structure

$$\begin{pmatrix} z & y + b_1w^{l_1} & a_1x^2 + a_2xw + b_2w^{l_2} \\ w^k & x & y \end{pmatrix}$$

where $a_1 \in \mathbb{C}\{x, w\}$, $a_2, b_1, b_2 \in \mathbb{C}\{w\}$ and at least one of $a_1(0)$ and $a_2(0)$ and one of $b_1(0)$ and $b_2(0)$ non-zero. If $a_2(0) \neq 0$, we can cancel the other term by an appropriate coordinate change in w (and, of course, cleaning up as before) and obtain the second series, because the term of higher relative matrix weight out of $b_1w^{l_1}$ and $b_2w^{l_2}$ can be killed by determinacy. If $a_2(0) = 0$, $l_1 = 2$ and $b_1(0) \neq 0$, we can cancel $b_2w^{l_2}$ again due to determinacy and obtain the third series. If $a_2(0) = 0$, no monomial w^2 appears in $b_1w^{l_1}$, $l_2 = 3$ and $b_2(0) \neq 0$, then we obtain the 4th series. Otherwise, i.e. if $a_2(0) = 0$, no w^2 term appears in $b_1w^{l_1}$ and no w^3 term in $b_2w^{l_2}$, then the singularity cannot be simple as it is adjacent to the non-simple singularity ($J^{(3,2)}$, line 2).

The proof of simplicity can be done as in $J^{(4,4)}$ leading to the following possible adjacencies: D -series, A -series, $J^{(3,1)}$ (with w^2 in the upper right hand entry), $J^{(4,1)}$, $J^{(4,2)}$, $J^{(4,3)}$ and $J^{(4,4)}$. As not all singularities of series $J^{(4,4)}$ are simple, we need to consider these adjacencies more closely: A simple explicit calculation shows that these adjacencies, which are obtained by perturbing with w in the lower left-hand entry, only allow adjacencies to the first series of $J^{(4,4)}$ for singularities of the first series, to the very first singularity in the same series for the second series and to the singularity of Tjurina number 7 resp. 8 ($J^{(4,4)}$, line 4 resp. 5) for the last remaining series. All the above mentioned singularities of type $J^{(4,4)}$ are simple which in turn implies that the 3 series are simple as well.

$J^{(3,3)}$: A matrix with this 1-jet is of the structure

$$\begin{pmatrix} y & z & \alpha \\ x & y + \beta & z + \gamma \end{pmatrix}$$

where $\alpha, \beta, \gamma \in \mathfrak{m}^2$. By the table of non-simple singularities, we know from the table of non-simple singularities ($J^{(4,4)}$, line 1) that the generic matrix with the weights $\underline{a} = (1, 2, 3, 2)$ and

$$D = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \end{pmatrix}$$

cannot be simple. This implies, that simple singularities can only occur in the following 6 cases

- (a) $\alpha = xw + w^k$

- (b) $\alpha = xw, \gamma = w^k$
- (c) $\alpha = xw, \beta = w^k$
- (d) $\alpha = w^2, \gamma = x^2$
- (e) $\alpha = x^3 + w^2$
- (f) $\alpha = x^2, \gamma = w^2$

It can be seen directly by the usual determinacy argument that, whenever β resp. γ are not mentioned among the conditions, their terms which do not lead to another previously mentioned case can be cancelled in all of these cases.

To prove simplicity is rather easy for cases (d)-(f), because there are no non-simple singularities of sufficiently small Tjurina number. For cases (a)-(c), we obviously have adjacency relations (c) adjacent to (b) with the same k , (b) to (a) again with the same k and (a) adjacent to (c) with a drop in k by one. By lengthy, but explicit calculations (using the 'adjacency'-relations among the 1-jets) one can then rule out any other adjacencies than to the series of 1-jet-types $J^{(4,1)}$ - $J^{(4,4)}$ and $J^{(3,1)}$. As the latter ones are all simple, this implies simplicity.

$J^{(3,4)}$ - $J^{(3,6)}$: By the same kind of argument as for $J^{(4,5)}$, no simple singularities are possible.

□

Remark 3.4 *As one can see in the last column of the table, we found precisely the rational triple point singularities classified by Tjurina (see [Tju]), whose notation for their types is used. As the last three cases do not have a name in the article of Tjurina, we simply stated them in the same order as they appear there.*

3.4 Singularities in $(\mathbb{C}^5, 0)$

In this case we only need to consider matrices whose 1-jet involves at least 4 variables. The methods are basically the same as in the previous case, with one exception: For the case $J^{(5,2)}$, the problem of classification and of finding adjacencies can be reduced to the corresponding problem for plane curve singularities and deformations with sections thereof.

Theorem 3.5 *The simple isolated Cohen-Macaulay codimension 2 singularities in $(\mathbb{C}^5, 0)$ are the following ones:*

Jet-Type	Type	Presentation Matrix		τ
$J^{(5,1)}$	A_0^\sharp	$\begin{pmatrix} x & y & z \\ w & v & x \end{pmatrix}$		1

$J^{(5,2)}$	A_k^\sharp	$\begin{pmatrix} x & y & z \\ w & v & x^{k+1} + y^2 \end{pmatrix}$	$k \geq 1$	$k + 2$
	D_k^\sharp	$\begin{pmatrix} x & y & z \\ w & v & xy^2 + x^{k-1} \end{pmatrix}$	$k \geq 4$	$k + 2$
	E_6^\sharp	$\begin{pmatrix} x & y & z \\ w & v & x^3 + y^4 \end{pmatrix}$		8
	E_7^\sharp	$\begin{pmatrix} x & y & z \\ w & v & x^3 + xy^3 \end{pmatrix}$		9
	E_8^\sharp	$\begin{pmatrix} x & y & z \\ w & v & x^3 + y^5 \end{pmatrix}$		10
$J^{(4,1)}$	Π_k	$\begin{pmatrix} w & y & x \\ z & w & y + v^k \end{pmatrix}$	$k \geq 2$	$2k - 1$
$J^{(4,2)}$		$\begin{pmatrix} w & y & x \\ z & w & y^k + v^2 \end{pmatrix}$	$k \geq 2$	$k + 2$
		$\begin{pmatrix} w & y & x \\ z & w & yv + v^k \end{pmatrix}$		$2k$
		$\begin{pmatrix} w + v^k & y & x \\ z & w & yv \end{pmatrix}$		$2k + 1$
		$\begin{pmatrix} w + v^2 & y & x \\ z & w & y^2 + v^k \end{pmatrix}$		$k + 3$
		$\begin{pmatrix} w & y & x \\ z & w & y^2 + v^3 \end{pmatrix}$		7
$J^{(4,3)}$		$\begin{pmatrix} v^2 + w^k & y & x \\ z & w & v^2 + y^l \end{pmatrix}$	$l \geq k \geq 2$	$k + l + 1$
		$\begin{pmatrix} v^2 + w^k & y & x \\ z & w & yv \end{pmatrix}$	$k \geq 2$	$k + 4$
		$\begin{pmatrix} v^2 + w^k & y & x \\ z & w & y^2 + v^l \end{pmatrix}$	$k \geq 2, l \geq 3$	$k + l + 2$
		$\begin{pmatrix} wv + v^k & y & x \\ z & w & yv + v^k \end{pmatrix}$	$k \geq 3$	$2k + 1$
		$\begin{pmatrix} wv + v^k & y & x \\ z & w & yv \end{pmatrix}$	$k \geq 3$	$2k + 2$
		$\begin{pmatrix} wv + v^3 & y & x \\ z & w & y^2 + v^3 \end{pmatrix}$		8
		$\begin{pmatrix} wv & y & x \\ z & w & y^2 + v^3 \end{pmatrix}$		9
		$\begin{pmatrix} w^2 + v^3 & y & x \\ z & w & y^2 + v^3 \end{pmatrix}$		9
$J^{(4,4)}$		$\begin{pmatrix} z & y & x \\ x & w & v^2 + y^2 + z^k \end{pmatrix}$	$k \geq 2$	$k + 4$
		$\begin{pmatrix} z & y & x \\ x & w & v^2 + yz + y^k w \end{pmatrix}$	$k \geq 1$	$2k + 5$
		$\begin{pmatrix} z & y & x \\ x & w & v^2 + yz + y^{k+1} \end{pmatrix}$	$k \geq 2$	$2k + 4$

		$\begin{pmatrix} z & y & x \\ x & w & v^2 + yw + z^2 \end{pmatrix}$		8
		$\begin{pmatrix} z & y & x \\ x & w & v^2 + y^3 + z^2 \end{pmatrix}$		9
		$\begin{pmatrix} z & y & x + v^2 \\ x & w & vy + z^2 \end{pmatrix}$		7
		$\begin{pmatrix} z & y & x + v^2 \\ x & w & vz + y^2 \end{pmatrix}$		8
		$\begin{pmatrix} z & y & x + v^2 \\ x & w & y^2 + z^2 \end{pmatrix}$		9

Proof: As in the case of 4 variables, we need to check all possible 1-jets and exclude the non-simple singularities. To this end, we start again by giving the list of those non-simple singularities which we will need in the proof:

Jet-Type	Presentation Matrix	\underline{a}	D	τ
$J^{(5,2)}$	$\begin{pmatrix} x & y & z \\ w & v & x^4 + y^4 + \alpha \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 4 \\ 1 & 1 & 4 \end{pmatrix}$	$(1 \ 1 \ 4 \ 1 \ 1)$	11
	$\begin{pmatrix} x & y & z \\ w & v & x^3 + y^6 + \alpha \end{pmatrix}$	$\begin{pmatrix} 2 & 1 & 6 \\ 2 & 1 & 6 \end{pmatrix}$	$(2 \ 1 \ 6 \ 2 \ 1)$	12
$J^{(4,2)}$	$\begin{pmatrix} w + v^2 & y & x \\ z & w & y^3 + v^3 \end{pmatrix}$	$\begin{pmatrix} 2 & 1 & 2 \\ 3 & 2 & 3 \end{pmatrix}$	$(2 \ 1 \ 3 \ 2 \ 1)$	8
	$\begin{pmatrix} w + v^3 & y & x \\ z & w & y^2 + v^4 \end{pmatrix}$	$\begin{pmatrix} 3 & 2 & 3 \\ 4 & 3 & 4 \end{pmatrix}$	$(3 \ 2 \ 4 \ 3 \ 1)$	9
$J^{(4,4)}$	$\begin{pmatrix} z & y & x \\ x & w & v^2 + y^3 + z^3 (+yz^2 + yw) \end{pmatrix}$	$\begin{pmatrix} 2 & 2 & 4 \\ 4 & 4 & 6 \end{pmatrix}$	$(4 \ 2 \ 2 \ 4 \ 3)$	11
	$\begin{pmatrix} z & y & x \\ x & w & v^3 + y^2 + z^2 \end{pmatrix}$	$\begin{pmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \end{pmatrix}$	$(4 \ 3 \ 2 \ 5 \ 2)$	13
	$\begin{pmatrix} z & y & x \\ x & w & v^3 + y^3 + z^2 \end{pmatrix}$	$\begin{pmatrix} 6 & 4 & 9 \\ 9 & 7 & 12 \end{pmatrix}$	$(9 \ 4 \ 6 \ 7 \ 4)$	17
	$\begin{pmatrix} z & y & x \\ x & w & v^2 + y^4 + z^2 \end{pmatrix}$	$\begin{pmatrix} 2 & 1 & 3 \\ 3 & 2 & 4 \end{pmatrix}$	$(3 \ 1 \ 2 \ 2 \ 2)$	12
	$\begin{pmatrix} z & y & x + v^2 \\ x & w & vz + yz + vw \end{pmatrix}$	$\begin{pmatrix} 3 & 2 & 4 \\ 4 & 3 & 5 \end{pmatrix}$	$(4 \ 2 \ 3 \ 3 \ 2)$	10
	$\begin{pmatrix} z & y & x + v^3 \\ x & w & vy + z^2 \end{pmatrix}$	$\begin{pmatrix} 2 & 3 & 3 \\ 3 & 4 & 4 \end{pmatrix}$	$(3 \ 3 \ 2 \ 4 \ 1)$	9
	$\begin{pmatrix} z & y & x + v^3 \\ x & w & y^2 + yz + z^2 \end{pmatrix}$	$\begin{pmatrix} 2 & 2 & 3 \\ 3 & 3 & 4 \end{pmatrix}$	$(3 \ 2 \ 2 \ 3 \ 1)$	15
	$\begin{pmatrix} z & y & x + v^2 \\ x & w & vy + yz + z^3 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 3 & 3 \end{pmatrix}$	$(2 \ 2 \ 1 \ 3 \ 1)$	8
$J^{(4,5)}$	$\begin{pmatrix} x & y & \alpha \\ z & w & \beta \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}$	$(1 \ 1 \ 1 \ 1 \ 1)$	17

$J^{(4,6)}$	$\begin{pmatrix} x & y & z \\ w & \alpha & \beta \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}$	$(1 \ 1 \ 1 \ 2 \ 1)$	13
-------------	---	--	-----------------------	----

$J^{(5,1)}$: The matrix $j_1 M$ already defines an isolated singularity and is 1-determined, that is we may w.l.o.g assume that $M = j_1 M$. M can be deformed to

$$M' = \begin{pmatrix} x + \varepsilon & y & v \\ z & w & x \end{pmatrix}$$

with $\varepsilon \in \mathbb{C}$. Since the ideal defining the singular locus of M' is $\langle \varepsilon^2, \varepsilon v, \varepsilon w, \varepsilon z, \dots \rangle$, the singular locus is empty, M' is smooth and hence M simple. Because M is contact-equivalent to

$$\begin{pmatrix} x & y & v \\ z & w & x + y^2 \end{pmatrix},$$

we will call it A_0^+ ; the reason for this will become clear in the subsequent case.

$J^{(5,2)}$: Any matrix of type $J^{(5,2)}$ is contact-equivalent to a matrix

$$M \sim_C \begin{pmatrix} x & y & v \\ z & w & f(x, y) \end{pmatrix}$$

with $f(x, y) \in \mathfrak{m}^2$. We will show that the properties and the behaviour of the singularity defined by M are determined by the hypersurface singularity defined by $f(x, y)$ in $\mathbb{C}\{x, y\}$.

Singular locus: By direct computation, the singular locus of M is completely contained in the plane defined by $\langle z, v, w \rangle$, and in this plane, the singular locus of M contains exactly the same points as the singular locus of the singularity defined by f .

Contact-Equivalence: If $f \in \mathbb{C}\{x, y\}$ is contact-equivalent to some $g \in \mathbb{C}\{x, y\}$, there is an isomorphism γ of $\mathbb{C}\{x, y\}$ and an unit $u \in \mathbb{C}\{x, y\}$ such that $ug = f \circ \gamma$. If γ is given by $\gamma(x) = \alpha_1 x + \beta_1 y$ and $\gamma(y) = \alpha_2 x + \beta_2 y$ with $\alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{C}\{x, y\}$, we can extend γ to an isomorphism of $\mathbb{C}\{x, y, z, w, v\}$ by defining $\gamma(v) = uv$, $\gamma(z) = \alpha_1 z + \beta_1 w$ and $\gamma(w) = \alpha_2 z + \beta_2 w$. This is an isomorphism showing $M = \begin{pmatrix} x & y & v \\ z & w & f(x, y) \end{pmatrix}$ is contact-equivalent to $\begin{pmatrix} x & y & v \\ z & w & g(x, y) \end{pmatrix}$.

Adjacencies: By direct computation, $T^1(M)$ can contain only elements of the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & h(x, y) \end{pmatrix}$$

with $h \in \mathbb{C}\{x, y\} / \left(f, x \frac{\partial f}{\partial x}, y \frac{\partial f}{\partial x}, x \frac{\partial f}{\partial y}, y \frac{\partial f}{\partial y} \right) = \mathbb{C}\{x, y\} / (f, \mathfrak{m}J(f))$. This is just the T^1 with section of the hypersurface singularity defined by f , and hence the adjacencies of M are determined by the adjacencies of f .

In this way we get the simple isolated singularities $A_k^+, D_k^+, E_6^+, E_7^+$ and E_8^+ . All of them can be deformed into the singularity A_0^+ we will get in case $J^{(5,1)}$ corresponding to the smooth curve $f(x, y) = x + y^2$, A_0 .

$J^{(4,1)}$: Any matrix M of type $J^{(4,1)}$ is contact-equivalent to a matrix

$$\begin{pmatrix} x & y & z + \delta_1 v^{k_1} \\ z & w & y + \delta_2 v^{k_2} \end{pmatrix}$$

with $\delta_1, \delta_2 \in \{0, 1\}$ and $k_1, k_2 \geq 2$. If $\delta_1 = \delta_2 = 0$, the singularity defined by M is not isolated. If $\delta_1 \neq 0$ or $\delta_2 \neq 0$, M is contact-equivalent to a matrix

$$\begin{pmatrix} x & y & z \\ z & w & y + v^k \end{pmatrix}$$

with $k = \min\{k_i | \delta_i \neq 0\}$. (If $k_1 = k_2$, one of the two terms may be cancelled by a lengthy sequence of coordinate changes and row and column operations.)

We will call the singularity defined by M Π_k . The singularity Π_k can only be deformed to

$$M' \sim_c \begin{pmatrix} x & y & z + \alpha_{k-1} v^{k-1} + \dots + \alpha_1 v + \alpha_0 \\ z & w & y + v^k + \beta_{k-2} v^{k-2} + \dots + \beta_1 v + \beta_0 \end{pmatrix}$$

with $\alpha_i, \beta_j \in \mathbb{C}$, which is contact-equivalent to

$$\begin{pmatrix} x & y & z \\ z & w & y + v^{k'} \end{pmatrix}$$

with $k' < k$. For $k' = 0$, M' is smooth, for $k' = 1$, it is contact-equivalent to the singularity A_0^+ , and for $k' > 1$, it is just $\Pi_{k'}$.

$J^{(4,2)}$: A matrix of this type is contact-equivalent to a matrix of the structure

$$\begin{pmatrix} w + \alpha & y & x \\ z & w & \beta \end{pmatrix}$$

where $\alpha, \beta \in \mathfrak{m}^2$, α only involving terms in v , β only involving terms in y and v . By the table of non-simple singularities ($J^{(4,2)}$, line 1), simple singularities cannot occur if the order of β is at least 3. Therefore we may assume (after a coordinate change in v and cleaning up w by column operations) that the 2-jet of β is one of the following: $y^2 + v^2$, v^2 , yv and y^2 . In the first case, all terms in α and the terms of higher order in β

may be cancelled due to determinacy, and we obtain the first matrix of the first series. If the 2-jet of β is v^2 , we can get rid of all terms of α by subtracting appropriate multiples of the last column from the second one and then cleaning up by a sequence of coordinate changes in y , w and z and a row operation. Since an appropriate coordinate change in v cancels all higher order terms in β which involve v , the matrix in this case is (by determinacy) of the structure

$$\begin{pmatrix} w & y & x \\ z & w & v^2 + y^k \end{pmatrix}$$

for some $k > 2$, which is the first series. If the term v^2 is not present, but the term yv occurs, then all terms of higher order which involve y may be cancelled by a coordinate change in v (and possibly cleaning up again). This provides us with a matrix

$$\begin{pmatrix} w + a_1 v^{k_1} & y & x \\ z & w & yv + a_2 v^{k_2} \end{pmatrix}$$

where $a_1, a_2 \in \mathbb{C}\{v\}$ either zero or a unit. By determinacy we then obtain the second series. In the last of the four cases, we see from the table of non-simple singularities ($J^{(4,2)}$, line 2) that the singularity cannot be simple if α is of order at least 3 and the 3-jet of β does not involve v^3 . In other words, we can only have further candidates for simple singularities, if the matrix is of the structure

$$\begin{pmatrix} w + v^2 & y & x \\ z & w & y^2 + \gamma \end{pmatrix}$$

or

$$\begin{pmatrix} w + \alpha & y & x \\ z & w & y^2 + v^3 \end{pmatrix}$$

where $\alpha, \gamma \in \mathfrak{m}^3$, α only involving v and γ only involving terms v^l and yv^l . The terms yv^l of γ may be cancelled by appropriate coordinate changes and row and column operations, since $l \geq 2$; this gives rise to the third series. Again by sequence of column operations and coordinate changes in w and z , we can get rid of α , providing the last singularity for this case. To proof simplicity, we can apply the same reasoning as in the case of 4 variables and obtain that, in addition to the adjacencies which preserve the 1-jet of the matrix, adjacencies are at most possible to singularities of the A -series, of 1-jet $J^{(4,1)}$, $J^{(5,1)}$ and $J^{(5,2)}$. For adjacencies preserving the 1-jet of the matrix, we may again follow the classification to see that adjacencies to the non-simple singularities of this 1-jet are impossible. A direct but rather lengthy calculation (relying on the fact that all entries are of order at most 2 in the original matrix and tracing this fact throughout the whole computation) then shows that no adjacencies to non-simple singularities of type $J^{(5,2)}$ are possible.

$J^{(4,3)}$: In this case, the matrix is of the structure ⁴

$$\begin{pmatrix} \alpha & y & x \\ z & w & \beta \end{pmatrix}$$

where $\alpha, \beta \in \mathfrak{m}^2$, α only involving w and v , β only involving y and v . By the table of non-simple singularities ($J^{(4,2)}$, line 2), we see immediately that no simple singularities can occur if the order of α or β exceeds 2. As α and β both have to be of order 2, it turns out to be a suitable approach to consider three cases:

- (a) α and β contain a term v^2

By appropriate coordinate changes in v followed by applying column operations of type 'addition of monomial times the second column to the first' resp. 'to the third column' and suitable coordinate changes in x and z , we can obtain a matrix of the following structure after applying the usual determinacy argument:

$$\begin{pmatrix} v^2 + w^k & y & x \\ z & w & v^2 + y^l \end{pmatrix}$$

These are exactly the matrices of the first series.

- (b) only one of α and β contains a term v^2

W.l.o.g. we may assume that the term v^2 appears in α . By the same transformation as in the case (a), the matrix is then of the structure

$$\begin{pmatrix} v^2 + w^k & y & x \\ z & w & \beta \end{pmatrix}$$

where β only involves v and y , is of order 2 and does not contain the term v^2 . Therefore the 2-jet of β is of the form $ay^2 + byv$. If $b \neq 0$, we can kill the term ay^2 as well as all higher pure powers of y by a suitable coordinate change in v (and of course subsequent cleaning up in the upper left hand entry). By applying a determinacy argument the matrix is then of the structure

$$\begin{pmatrix} v^2 + w^k & y & x \\ z & w & yv \end{pmatrix}.$$

If $b = 0$, β is of the structure $\tilde{a}y^2 + cyv^2 + dv^l$ where $\tilde{a}, b, c \in \mathbb{C}\{y, v\}$, \tilde{a} and d units. We can get rid of the cyv^2 by an appropriate coordinate change in y , applying column operations of type 'addition of monomial times first column to the second' and subsequent cleanup by row and column operations and coordinate changes in x and z . After

⁴At this point it is important to observe that the roles of y and w may harmlessly be interchanged.

applying a determinacy argument, this provides us with a matrix of the structure

$$\begin{pmatrix} v^2 + w^k & y & x \\ z & w & y^2 + v^l \end{pmatrix}$$

where $l \geq 3$ which is exactly the third series.

- (c) neither α nor β contain a term v^2

By the same arguments as at the beginning of case (b), we need to distinguish between three cases: the 2-jets of α and β both still involve v , only one involves v or both do not involve v . In the first case, the 2-jet of the matrix is

$$\begin{pmatrix} wv & y & x \\ z & w & yv \end{pmatrix}$$

and the only higher order terms in α and β which cannot be cancelled are pure powers of v yielding exactly the 4th series.

If there is only one mixed term in the 2-jet, we may w.l.o.g. assume that it appears in α . By the same kind of sequence of row and column operations and coordinate changes as before we obtain a matrix of the structure

$$\begin{pmatrix} wv + av^3 & y & x \\ z & w & y^2 + bv^3 + cyv^2 \end{pmatrix}$$

where $a, b, c \in \mathbb{C}\{v\}$. If b is not a unit, perturbing the upper left hand entry with w leads to the non-simple singularity ($J^{(4,2)}$, line 2). By an appropriate coordinate change in v (and subsequent cleaning up), we may hence safely assume that $b = 1$ and $c = 0$. If a is a unit, we are dealing with the first of the three remaining singularities.

If a is not a unit, we can write av^3 as $\tilde{a}v^4$ and a coordinate change in w can move it to the middle entry on the bottom row where it appears as $\tilde{a}v^3$ which can in turn be cancelled by adding \tilde{a} times the third column to the second one (and subsequent cleaning up). This singularity is the second one of the remaining three.

In the final case, where α and β do not involve v , the only singularity which is not adjacent to the non-simple one already mentioned above is

$$\begin{pmatrix} w^2 + v^3 & y & x \\ z & w & y^2 + v^3 \end{pmatrix}$$

which is exactly the last singularity in the list.

The proof that these singularities are indeed simple involves the same kind of arguments as in the case $J^{(4,2)}$, allowing, in addition to adjacencies preserving the 1-jet, at most adjacencies to singularities of the A -series, of 1-jet $J^{(5,1)}$, $J^{(5,2)}$, $J^{(4,2)}$ and $J^{(4,1)}$. By basically the same lengthy calculations as in the case $J^{(4,2)}$, it can again be shown that adjacencies to the non-simple singularities of the same 1-jet, of 1-jet $J^{(4,2)}$ and of 1-jet $J^{(5,2)}$ cannot occur.

$J^{(4,4)}$: A matrix with 1-jet of type $J^{(4,4)}$ is of the structure

$$\begin{pmatrix} z & y & x + \alpha \\ x & w & \beta \end{pmatrix}$$

where $\alpha, \beta \in \mathfrak{m}^2$, α only involving v . From the table of non-simple singularities ($J^{(4,4)}$, line1), we see first of all that simple singularities can only occur if the order of β is 2 and by ($J^{(4,4)}$, all lines) that for simple singularities the 2-jet of β cannot consist of a single pure power. Moreover, direct computation of the singular locus shows that we need at least one pure power of v in α or β . This still leaves a rather large number of possibilities for the 2-jet of β which we can divide into four cases:

- (a) β contains the term v^2

By a suitable coordinate change in v , we may assume that v^2 is the only term in β which involves v . Using the fact that the 2-jet of β cannot consist of a single pure power, a direct calculation shows that the following 7 cases may occur (which we already saw in the case of 4 variables): $v^2 + y^2 + z^2$, $v^2 + yz + yw$, $v^2 + z^2 + yw$, $v^2 + y^2$, $v^2 + yz$, $v^2 + z^2$, $v^2 + yw$. In the first three cases the matrix is already 2-determined and due to weighted determinacy we can even get rid of all terms of α obtaining three singularities of which the first two are just the beginning of the first two series. The remaining cases give rise to two more series and two additional singularities by the same reasoning as in the case $J^{(4,4)}$ in 4 variables each time, of course, using the weighted determinacy to remove the terms of α .

- (b) β does not contain v^2 , but vy

In this case, direct calculation shows that we can have the following 2-jets of β : $vy + yz + z^2$, $vy + yz$, $vy + z^2$ and vy . If the order of α is at least three we can perturb a matrix in this case to the first of the 4 possibilities for β and to v^3 for α ; in this matrix the term yz of β is then killed by determinacy and we obtain the matrix $J^{(4,4)}$, line 6, from the table of non-simple singularities. Hence simple singularities can only occur if α contains the term v^2 . For determinacy reasons, the first and the third of the above 4 cases then coincide leading to the singularity

$$\begin{pmatrix} z & y & x + v^2 \\ x & w & vy + z^2 \end{pmatrix}.$$

The other two cases cannot lead to simple singularities according to the list of non-simple singularities.

- (c) β contains neither v^2 nor vy , but vz

Similar to the previous case, the only possible 2-jets of β are $vz + y^2 + yz$, $vz + y^2$, $vz + yz$ and vz . As perturbing β with the term vy takes us to the case (b), we see that α can be of order at most 2. By determinacy, the first and second case then give rise to the same

singularity

$$\begin{pmatrix} z & y & x + v^2 \\ x & w & vz + y^2 \end{pmatrix}.$$

The other two cases are adjacent (by the perturbation term vy) to the non-simple singularity mentioned in (b).

- (d) β does not contain any terms involving v

For the 2-jet of β , a direct calculation with row and column operations and coordinate changes in y and w shows that there are not many possibilities left in this case: $y^2 + z^2$, y^2 , yz and z^2 . In all of these cases, the singularities are adjacent to singularities from case (b) which implies that α has to be of order 2 for simple singularities. Moreover, the second case is adjacent to the non-simple singularity from case (b) (by perturbation with vy) and the last two are adjacent to non-simple singularities from case (c) (by perturbation with vz). This only leaves the first case, namely

$$\begin{pmatrix} z & y & x + v^2 \\ x & w & y^2 + z^2 \end{pmatrix},$$

which is the last singularity in the list stated in the theorem.

For proving simplicity of these singularities, it is first of all important to show that the singularities, which we found as candidates in this case $J^{(4,4)}$, cannot be adjacent to any of the non-simple singularities of the same case. Due to the large number of cases, this is a lengthy, but straightforward calculation. On the other hand, adjacencies to singularities with other 1-jets of the matrix can be checked explicitly, which leads to possible adjacencies to the D-series, the A-series, singularities of 1-jet $J^{(4,1)}$, $J^{(4,2)}$, $J^{(4,3)}$, $J^{(5,1)}$ and $J^{(5,2)}$. The main ingredients to computing directly that adjacencies to non-simple singularities of types $J^{(5,2)}$, $J^{(4,3)}$ and $J^{(4,2)}$ cannot occur, are the knowledge about the low order of α and β and about the structure of β in each of the cases; i.e. the 4 cases (a) to (d) are considered separately which makes the calculation even lengthier than the previous ones.

$J^{(4,5)}$: According to the table of non-simple singularities the generic matrix of this 1-jet cannot be simple.

$J^{(4,6)}$: Again there cannot be any simple singularities according to the table of non-simple singularities.

□

3.5 Singularities in $(\mathbb{C}^6, 0)$

In this case, we can only exclude the 1-jets containing 3 variables which implies that we have to consider jets containing 4, 5 and 6 variables. Fortunately, there

turns out to be only one case containing 6 variables and only very few cases with 4 variables contributing to the list of simple singularities. Parts of the proof parallel the classifications of simple hypersurface singularities of dimension 2, other parts rely on the one of simple fat point singularities in the plane.

Theorem 3.6 *The simple isolated Cohen-Macaulay codimension 2 singularities in $(\mathbb{C}^6, 0)$ are listed in the following table:.*

Jet-Type	Type	Presentation Matrix		τ
$J^{(6,1)}$	Ω_1	$\begin{pmatrix} x & y & v \\ z & w & u \end{pmatrix}$		0
$J^{(5,1)}$	Ω_k	$\begin{pmatrix} x & y & v \\ z & w & x + u^k \end{pmatrix}$	$k \geq 2$	$k - 1$
$J^{(5,2)}$	$A_k^\#$	$\begin{pmatrix} x & y & z \\ w & v & u^2 + x^{k+1} + y^2 \end{pmatrix}$	$k \geq 1$	$k + 2$
	$D_k^\#$	$\begin{pmatrix} x & y & z \\ w & v & u^2 + xy^2 + x^{k-1} \end{pmatrix}$	$k \geq 4$	$k + 2$
	$E_6^\#$	$\begin{pmatrix} x & y & z \\ w & v & u^2 + x^3 + y^4 \end{pmatrix}$		8
	$E_7^\#$	$\begin{pmatrix} x & y & z \\ w & v & u^2 + x^3 + xy^3 \end{pmatrix}$		9
	$E_8^\#$	$\begin{pmatrix} x & y & z \\ w & v & u^2 + x^3 + y^5 \end{pmatrix}$		10
		$\begin{pmatrix} x & y & z \\ w & v & ux + y^k + u^l \end{pmatrix}$	$k \geq 2, l \geq 3$	$k + l - 1$
		$\begin{pmatrix} x & y & z \\ w & v & x^2 + y^2 + u^3 \end{pmatrix}$		6
$J^{(4,1)}$	$F_{q,r}^\#$	$\begin{pmatrix} w & y & x \\ z & w + vu & y + v^q + u^r \end{pmatrix}$	$q, r \geq 2$	$q + r$
	$G_5^\#$	$\begin{pmatrix} w & y & x \\ z & w + v^2 & y + u^3 \end{pmatrix}$		7
	$G_7^\#$	$\begin{pmatrix} w & y & x \\ z & w + v^2 & y + u^4 \end{pmatrix}$		10
	$H_{q+3}^\#$	$\begin{pmatrix} w & y & x \\ z & w + v^2 + u^q & y + vu^2 \end{pmatrix}$	$q \geq 3$	$q + 5$
	$I_{2q-1}^\#$	$\begin{pmatrix} w & y & x \\ z & w + v^2 + u^3 & y + u^q \end{pmatrix}$	$q \geq 4$	$2q + 1$
	$I_{2r+2}^\#$	$\begin{pmatrix} w & y & x \\ z & w + v^2 + u^3 & y + vu^r \end{pmatrix}$	$r \geq 3$	$2r + 4$
$J^{(4,2)}$		$\begin{pmatrix} w & y & x \\ z & w + v^{k_1} + u^{k_2} & y^l + uv \end{pmatrix}$	$k_1, k_2, l \geq 2$	$k_1 + k_2 + l - 1$
		$\begin{pmatrix} w & y & x \\ z & w + v^2 & u^2 + yv \end{pmatrix}$		6
		$\begin{pmatrix} w & y & x \\ z & w + uv & u^2 + yv + v^k \end{pmatrix}$	$k \geq 3$	$k + 4$

		$\begin{pmatrix} w & y & x \\ z & w + v^k & u^2 + yv + v^3 \end{pmatrix}$	$k \geq 3$	$2k + 2$
		$\begin{pmatrix} w & y & x \\ z & w + uv^k & u^2 + yv + v^3 \end{pmatrix}$	$k \geq 2$	$2k + 5$
		$\begin{pmatrix} w & y & x \\ z & w + v^3 & u^2 + yv \end{pmatrix}$		9
		$\begin{pmatrix} w & y & x \\ z & w + v^k & u^2 + y^2 + v^3 \end{pmatrix}$	$k \geq 3$	$2k + 3$
		$\begin{pmatrix} w & y & x \\ z & w + uv^k & u^2 + y^2 + v^3 \end{pmatrix}$	$k \geq 2$	$2k + 6$

Proof: First of all, we state a table of non-simple singularities in 6 variables:

Jet-Type	Presentation Matrix	\underline{a}	D	τ
$J^{(5,2)}$	$\begin{pmatrix} x & y & z \\ w & v & u^2 + x^3 + y^6 \end{pmatrix}$	$\begin{pmatrix} 2 & 1 & 6 \\ 2 & 1 & 6 \end{pmatrix}$	$(2 \ 1 \ 6 \ 2 \ 1 \ 3)$	12
	$\begin{pmatrix} x & y & z \\ w & v & u^2 + x^4 + y^4 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 4 \\ 1 & 1 & 4 \end{pmatrix}$	$(1 \ 1 \ 4 \ 1 \ 1 \ 2)$	11
	$\begin{pmatrix} x & y & z \\ w & v & x^2 + y^2 + \alpha \end{pmatrix}$	$\begin{pmatrix} 2 & 2 & 5 \\ 1 & 1 & 4 \end{pmatrix}$	$(2 \ 2 \ 5 \ 1 \ 1 \ 1)$	8
	$\begin{pmatrix} x & y & z \\ w & v & y^2 + x^3 + u^3 \end{pmatrix}$	$\begin{pmatrix} 2 & 3 & 7 \\ 1 & 2 & 6 \end{pmatrix}$	$(2 \ 3 \ 7 \ 1 \ 2 \ 2)$	8
$J^{(4,1)}$	$\begin{pmatrix} w & y & x \\ z & w + u^2 & y + v^5 \end{pmatrix}$	$\begin{pmatrix} 4 & 5 & 6 \\ 3 & 4 & 5 \end{pmatrix}$	$(6 \ 5 \ 3 \ 4 \ 1 \ 2)$	13
	$\begin{pmatrix} w & y & x \\ z & w + u^3 & y + v^3 \end{pmatrix}$	$\begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix}$	$(3 \ 3 \ 3 \ 3 \ 1 \ 1)$	12
$J^{(4,2)}$	$\begin{pmatrix} w & y & x \\ z & w + \alpha & \beta \end{pmatrix}$	$\begin{pmatrix} 4 & 3 & 3 \\ 5 & 4 & 4 \end{pmatrix}$	$(3 \ 3 \ 5 \ 4 \ 1 \ 2)$	11
	$\begin{pmatrix} w & y & x \\ z & w + \alpha & \beta \end{pmatrix}$	$\begin{pmatrix} 2 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix}$	$(3 \ 2 \ 2 \ 2 \ 1 \ 1)$	10
	$\begin{pmatrix} w & y & x \\ z & w + \alpha & \beta \end{pmatrix}$	$\begin{pmatrix} 6 & 2 & 4 \\ 6 & 4 & 6 \end{pmatrix}$	$(4 \ 2 \ 6 \ 4 \ 2 \ 3)$	9

$J^{(6,1)}$: By determinacy a singularity with this 1-jet is of the form

$$\begin{pmatrix} x & y & v \\ z & w & u \end{pmatrix}.$$

Since $T^1(M) = 0$, this singularity is rigid and in particular simple.

$J^{(5,1)}$: In this case, the matrix is of the structure

$$\begin{pmatrix} x & y & v \\ z & w & x + u^j \end{pmatrix}$$

for some $j \geq 2$. Calculating $T^1(M)$, we see that the versal deformation of M is of the form

$$\begin{pmatrix} x & y & v \\ z & w & x + u^j + \sum_{i=0}^{j-2} \alpha_i u^i \end{pmatrix}$$

where $\alpha_i \in \mathbb{C}$ and not all $\alpha_i = 0$. At each zero of $u^j + \sum_{i=0}^{j-2} \alpha_i u^i$, this matrix is contact-equivalent to the rigid singularity Ω_1 or to a germ Ω_k for some $k < l$. Thus all Ω_j for $j \geq 2$ are simple.

$J^{(5,2)}$: Here the matrix is of the structure

$$\begin{pmatrix} x & y & z \\ w & v & \alpha \end{pmatrix}$$

where $\alpha \in \mathfrak{m}^2$, not involving z, w and v . In contrast to $J^{(5,2)}$ in 5 variables, we can not directly reduce this case to the classification of simple isolated hypersurface singularities in 3 variables, because the variable u which does not appear in the 1-jet of the matrix plays a different role than the variables x and y . Thus we will distinguish between three cases⁵ depending on the way u appears in the 2-jet of α :

- (a) the 2-jet of α contains u^2

In this case, the other terms containing u may be cancelled directly by an appropriate coordinate change in u leading to a matrix

$$\begin{pmatrix} x & y & z \\ w & v & u^2 + \beta \end{pmatrix}$$

where $\beta \in \mathfrak{m}^2$, only involving x and y . By the same calculations as in the case $J^{(5,2)}$ in 5 variables, it can now be shown that the singular locus corresponds to $(z, v, w, u, \beta, \frac{\partial \beta}{\partial x}, \frac{\partial \beta}{\partial y})$ implying that points of the singular locus correspond exactly to the points of the one of β . The reformulation of contact equivalence and T^1 also lead to the same observations as in the case of 5 variables. Therefore we obtain simple singularities from this case exactly for β being an E_6, E_7, E_8, D_k or A_k singularity.

- (b) the 2-jet of α contains ux , but not u^2

Here we can apply an appropriate coordinate change in x (followed by cleaning the first column by a suitable column operation of type

⁵Note that the roles of x and y can harmlessly be exchanged which explains why we can assume in the second case that the mixed term is ux .

'adding the second column to the first' and a subsequent coordinate change in w) and hence assume that there are no terms in α which are divisible by yu . A further coordinate change, this time in u allows us to get rid of all terms containing the factor x except xu , of course, yielding a matrix of the structure

$$\begin{pmatrix} x & y & z \\ w & v & ux + ay^k + bu^l \end{pmatrix}$$

where $a \in \mathbb{C}\{y\}$, $b \in \mathbb{C}\{u\}$ are both units, because a matrix of this structure does not define an isolated singularity if there is no pure power of u or no pure power of y in the lower right hand entry. By determinacy, this is exactly the last series listed for this 1-jet in the table of the theorem.

- (c) the 2-jet of α does not contain any terms involving u

In this case, we see from the table of non-simple singularities ($J^{(5,2)}$, line 4) that a simple singularity may only occur if the 2-jet of α is not a square. This implies that we only need to consider matrices of the structure

$$\begin{pmatrix} x & y & z \\ w & v & x^2 + y^2 + \beta \end{pmatrix}$$

where $\beta \in \mathfrak{m}^3$. According to the table of non-simple singularities ($J^{(5,2)}$, line 3), all singularities in this case cannot be simple unless they contain the term u^3 . In this case determinacy implies that this is exactly the last singularity in the list for this 1-jet.

Since adjacencies to singularities of hypersurface types cannot occur in this case and since all singularities of 1-jet $J^{(6,1)}$ and $J^{(5,2)}$ are simple, we only need to consider adjacencies which do not change the 1-jet. In the case (a) all of these adjacency calculations are exactly analogous to the case of 2-dimensional hypersurfaces proving simplicity of the respective singularities. In case (b) the only adjacencies preserving the 1-jet are the ones into singularities of the same case and into A_k type of singularities of case (a), as a direct calculation shows; for the last singularity we can already deduce from the Tjurina number that adjacencies to non-simple singularities of the cases (a) and (b) are impossible.

$J^{(4,1)}$: A singularity of this kind corresponds to a matrix of the structure

$$\begin{pmatrix} w & y & x \\ z & w + \alpha & y + \beta \end{pmatrix}$$

where $\alpha, \beta \in \mathfrak{m}^2$, involving only the variables u and v . Similar to the case $J^{(5,2)}$ in 5 variables, the best approach to this case is to first consider the singular locus of these singularities. It turns out to be defined by $(x, y, z, w, \alpha, \beta)$ implying that the given matrix can only define an isolated singularity if (α, β) corresponds to a fat point in the plane. A rather

direct calculation shows that two singularities defined by matrices of this kind are contact-equivalent if and only if the corresponding fat points are contact-equivalent. Moreover, the T^1 can only contain elements of the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & a & b \end{pmatrix}$$

where

$$\begin{pmatrix} a \\ b \end{pmatrix} \in (\mathbb{C}\{u, v\}/(\alpha, \beta))^2 / \left(\left(\frac{\partial \alpha}{\partial u}, \frac{\partial \alpha}{\partial v} \right) \right).$$

This is exactly the T^1 of the corresponding fat point singularity. Therefore simple singularities with the given 1-jet of the matrix are exactly the ones of the above mentioned structure for which (α, β) is a fat point from the list of simple isolated complete intersection singularities which are not of type A_k .

$J^{(4,2)}$: For studying matrices in this case, the first important observation is that these matrices are adjacent to those of 1-jet $J^{(4,1)}$ by perturbing with y in the lower right hand entry. Writing a matrix of the given 1-jet as

$$\begin{pmatrix} w & y & x \\ z & w + \alpha & \beta \end{pmatrix}$$

we can hence deduce that this singularity can only be simple if at least one of α and β contains one of the order two terms uv and u^2 (or, of course, v^2 which coincides with the u^2 case by exchanging the roles of u and v which is still possible at this point).

(a) $j_2\beta$ not a square modulo y

By appropriate row and column operations and coordinate changes of u , v and z , we may assume that a matrix in this case is of the structure

$$\begin{pmatrix} w & y & x \\ z & w + a_1 u^{k_1} + a_2 v^{k_2} & uv + by^l \end{pmatrix}$$

for suitable k_1, k_2, l and $a_1 \in \mathbb{C}\{u\}$, $a_2 \in \mathbb{C}\{v\}$ and $b \in \mathbb{C}\{y\}$ units, because the singularity is no longer isolated if any of the three terms is missing. By determinacy, this leads to the first series of $J^{(4,2)}$.

(b) $j_2\beta = u^2$ modulo y

In this case, a coordinate change in u allows us to cancel the term yu in β and the term u^2 in α can be killed by a sequence of appropriate row and column operations followed by a coordinate change in z (and subsequent cleanup). Therefore the 2-jet of the matrix can be assumed to be of the form

$$\begin{pmatrix} w & y & x \\ z & w + auv + bv^2 & u^2 + cyv + dy^2 \end{pmatrix}.$$

where $a, b, c, d \in \mathbb{C}$. If $c \neq 0$, a coordinate change in v followed by one in w , a column operation of type 'adding second column to first column' and a coordinate change in z allows us to remove the term dy^2 . If in addition to that also $b \neq 0$, then determinacy yields that the matrix is

$$\begin{pmatrix} w & y & x \\ z & w + v^2 & u^2 + yv \end{pmatrix}.$$

Still in the case $c \neq 0$, but this time $b = 0$, $a \neq 0$, the situation is a little bit more difficult, since we cannot use determinacy to remove the higher order pure powers of v in α . Instead, we make a coordinate change in u which produces two kinds of terms in β : terms of the form uv^l which we will remove and pure powers in v which we will keep. For cancelling the terms uv^l , we first perform a column operation of type 'adding second column to the third' (and clean up the upper right hand entry by a coordinate change in x) replacing uv^l by $v^{l-1}w$. A coordinate change in y now allows us to move these terms to terms $v^{l-2}w$ in the upper middle entry and by a column operation of type 'adding first column to the second' to the bottom middle entry as $v^{l-2}z$. As l was at least three in our case, we can now shift these terms back to the lower right hand entry by a suitable coordinate change in u and eventually kill them by a column operation of type 'adding first column to third column'. By determinacy the matrix is then of the form

$$\begin{pmatrix} w & y & x \\ z & w + uv & u^2 + yv + v^k \end{pmatrix}$$

for a suitable $k \geq 3$ (If the v^k term were missing the singularity would not stand a chance to be isolated.)

Again still in the case $c \neq 0$, but now in the situation $a = b = 0$, we can conclude from the table of non-simple singularities ($J^{(4,2)}$, all lines) that a simple singularity can only occur if the term v^3 is present in at least one of α and β . If it is present in β , then the matrix is of the structure

$$\begin{pmatrix} w & y & x \\ z & w + \gamma_1 uv^{k_1} + \gamma_2 v^{k_2} & u^2 + yv + v^3 \end{pmatrix}$$

where $\gamma_1, \gamma_2 \in \mathbb{C}\{v\}$ and at least one of them a unit. By determinacy, exactly one term uv^{k_1} or v^{k_2} remains and we obtain the third series. If there is no v^3 in β , then it appears in α leading, by determinacy, to the matrix

$$\begin{pmatrix} w & y & x \\ z & w + v^3 & u^2 + yv \end{pmatrix}.$$

This ends the arguments in the case $c \neq 0$.

In the case $c = 0$, $d \neq 0$, simple singularities may only occur if the

term v^3 is present in β , since otherwise an adjacency to the non-simple singularity of jet type $J^{(5,2)}$, line 3, exists by perturbation of the middle entry in the bottom row by u . If v^3 is present in β , then the matrix is of the structure

$$\begin{pmatrix} w & y & x \\ z & w + a_1 uv^{k_1} + a_2 v^{k_2} & u^2 + y^2 + v^3 \end{pmatrix}$$

where $a_1, a_2 \in \mathbb{C}\{v\}$ either a unit or zero, at least one of them a unit. In contrast to the previous cases, we cannot proceed by the standard determinacy argument here, because some of these matrices are not quasihomogeneous in the strict sense, as x needs to be assigned non-positive weights if k_1 exceeds 2 and k_2 exceeds 4 in order to satisfy relative row and column weights. On the other hand, we can remove the higher order terms (w.r.t. the weights of $u^2 + y^2 + v^3$) by explicit calculation in the following way: If the lowest order term is of the form uv^{k_1} we start by a coordinate change in u killing $a_2 v^{k_2}$ which in turn introduces some $uv^2 \cdot p(v)$ into the lower left hand entry of which all terms except at most one are divisible by v^3 and can hence be collected into a coefficient of v^3 which is a unit. We can get rid of the last remaining term by a coordinate change in v such that all new terms can again be collected into the coefficients of u^2 and v^3 . By appropriate coordinate changes of y , u and w (multiplication by units), we can achieve that the matrix is of the form

$$\begin{pmatrix} e_1 w & e_2 y & x \\ z & e_1(w + uv^{k_1}) & e_3(u^2 + y^2 + v^3) \end{pmatrix}$$

where e_1, e_2, e_3 are units. These units can easily be removed by multiplication of rows and columns by units and subsequent coordinate changes in z and x . If, on the other hand, the lowest order term is of the form v^{k_2} , we start by a coordinate change in v and then remove the offending terms in the same way as before.

Finally, if $c = d = 0$ in the matrix

$$\begin{pmatrix} w & y & x \\ z & w + auv + bv^2 & u^2 + cyv + dy^2 \end{pmatrix}.$$

(from the beginning of this case) then the table of non-simple singularities ($J^{(4,2)}$, line 3) implies that there cannot be any simple singularities.

For proving simplicity, the rather straightforward explicit calculation shows in this situation that at most singularities of types A_k , $J^{(6,1)}$, $J^{(5,1)}$, $J^{(5,2)}$, $J^{(4,1)}$ and of course of the same 1-jet may occur. A_k , $J^{(6,1)}$ and $J^{(5,1)}$ do not contain non-simple singularities; that the non-simple ones of $J^{(5,2)}$, $J^{(4,1)}$ and $J^{(4,2)}$ cannot be reached can be shown directly by a calculation which relies on keeping track of the order of the entries in u and v .

(c) $j_2\beta = 0$ modulo y

In this case, the table of non-simple singularities ($J^{(4,2)}$, line 2) implies that no simple singularities can occur.

$J^{(4,3)}$ - $J^{(4,6)}$: According to the table of non-simple singularities, matrices with these 1-jets cannot define simple singularities.

□

3.6 Some Remarks on the Lists of Simple Singularities

In contrast to the table of simple singularities in 4 variables which does not show any unexpected behaviour, there are several surprising details about the tables in 5 and 6 variables:

- The simple hypersurface singularities reappear as entries of the matrix in the cases $J^{(5,2)}$. This can easily be understood as a consequence of the fact that in the other 5 entries no perturbation exist which cannot be shifted into the last entry. Moreover only those variables cannot be removed from this last entry by row and column operations which are neither an entry in the same row nor in the same column.
- The simple fat point singularities in the plane reappear as entries of the matrix in the case $J^{(4,1)}$ in 6 variables resp. the fat points on the line in the same case in 5 variables. As before the obvious reason is that there are exactly two entries where relevant perturbations can occur and that at these entries only two resp. one variables really matter for the perturbations.
- The most surprising fact is that in 6 variables there are simple singularities which are not quasihomogeneous in the strict sense, since one of the variables has to be of non-positive weight in order to satisfy relative row and column weights.

4 Singularities in $(\mathbb{C}^2, 0)$ - Fat Points

As mentioned at the beginning of the previous section, the usual counting argument does not provide a bound for the number of rows of a candidate in dimension 2. Therefore the process of finding candidates has to consider all possible matrix sizes. Thus we start by considering the 3×2 matrices, try to find weighted jets which do not allow simple singularities of this size and mark the remaining cases as candidates. Iterating this, we increase the matrix size step by step until we reach a matrix size for which we can prove that there cannot be any simple singularities (which luckily happens for the 4×3 matrices).

The following lemma will be the most important tool for proving that certain candidates are not simple in the case of fat points:

Lemma 4.1 *Quasihomogeneous 3×2 matrices w.r.t. the weights*

$$\left(\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}, (1 \ 1) \right)$$

respectively the weights

$$\left(\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}, (1 \ 1) \right)$$

cannot define a simple fat point singularity in $(\mathbb{C}^2, 0)$. The same statement holds for 4×3 matrices w.r.t. the weights

$$\left(\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \end{pmatrix}, (1 \ 1) \right)$$

respectively

$$\left(\begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{pmatrix}, (1 \ 1) \right).$$

Proof: In the first case, the total number of monomials which can appear in the matrix is $\sum_{ij} r(a, D_{ij}) = 15$ and the cardinalities of the sets S_i are $S_1 = 2$, $S_2 = 9$ and $S_3 = 4$ which shows that

$$\#S_1 + \#S_2 + \#S_3 - 2 = 13 < 15 = \sum_{ij} r(a, D_{ij}).$$

Therefore this singularity cannot be simple.
In the other cases the inequalities are

$$\#S_1 + \#S_2 + \#S_3 - 2 = 13 < 14 = \sum_{ij} r(a, D_{ij}),$$

$$\#S_1 + \#S_2 + \#S_3 - 2 = 25 < 28 = \sum_{ij} r(a, D_{ij}),$$

$$\#S_1 + \#S_2 + \#S_3 - 2 = 25 < 27 = \sum_{ij} r(a, D_{ij}).$$

□

As in the previous section, we now consider the 1-jets:

Lemma 4.2 *Let M be a 3×2 matrix with entries in the maximal ideal of $\mathbb{C}\{x, y\}$. Then $j_1 M$ is contact-equivalent to one of the two jets*

$$\begin{pmatrix} x & y & 0 \\ 0 & 0 & y \end{pmatrix} \text{ and } \begin{pmatrix} x & y & 0 \\ 0 & x & y \end{pmatrix}$$

or M cannot be simple.

Proof: Without loss of generality, we may assume that the matrix of j_1M is either of the form

$$\begin{pmatrix} x & y & 0 \\ * & * & * \end{pmatrix} \text{ or } \begin{pmatrix} x & 0 & 0 \\ * & * & * \end{pmatrix},$$

where the $*$ denotes an entry which may be zero or of degree 1.

In the **first** case, the last entry in the bottom row has to be non-zero, because otherwise M would not be simple by adjacency to a matrix of the second case of the previous lemma. By exchanging the roles of x and y and permuting the corresponding columns if necessary, we may assume that the 1-jet is

$$\begin{aligned} \begin{pmatrix} x & y & 0 \\ * & * & y + \alpha x \end{pmatrix} &\sim_C \begin{pmatrix} x & y - \alpha x & 0 \\ * & * & y \end{pmatrix} \\ &\sim_C \begin{pmatrix} x & y & 0 \\ \gamma_1 x & \gamma_2 x & y \end{pmatrix} \\ &\sim_C \begin{pmatrix} x & y & 0 \\ 0 & \gamma_2 x & y \end{pmatrix} \end{aligned}$$

where α, γ_1 and $\gamma_2 \in \mathbb{C}$. Moreover, we can assume at this point that $\gamma_2 \in \{0, 1\}$ by multiplying the last row and last column n by suitable constants if necessary.

If $\gamma_2 = 0$, M has the form of the first matrix in the statement of the lemma, of the second matrix otherwise.

In the **second** case from the beginning of the proof, we may again assume by the considerations from above that the 1-jet is

$$\begin{pmatrix} x & 0 & 0 \\ * & * & \alpha_1 x + \alpha_2 y \end{pmatrix}$$

$\alpha_2 \neq 0$: By direct computation, we obtain that the 1-jet is contact-equivalent to

$$\begin{pmatrix} x & 0 & 0 \\ 0 & \beta x & y \end{pmatrix}$$

where $\beta \in \{0, 1\}$. If $\beta = 0$, M cannot be simple by the previous lemma, otherwise we are in case 1 of the statement of this lemma.

$\alpha_2 = 0$: By direct computation, it turns out that this case is up to permutation of columns identical to case 1.

□

Lemma 4.3 *Let M be a 4×3 matrix with entries in the maximal ideal of $\mathbb{C}\{x, y\}$. Then M cannot be simple.*

Proof: After suitable row and column operations we may assume that j_1M is either of the form

$$\begin{pmatrix} x & \alpha y & 0 & 0 \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix} \text{ or } \begin{pmatrix} x & \alpha y & 0 & 0 \\ y & * & * & * \\ 0 & * & * & * \end{pmatrix},$$

where $\alpha \in \{0, 1\}$.

In the first case, let us consider the following perturbations of M resp. j_1M :

$$\begin{pmatrix} x & \alpha y & 0 & t \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix} \sim_{C \text{ of } 1\text{-jets}} \begin{pmatrix} 0 & 0 & 0 & t \\ 0 & * & * & 0 \\ 0 & * & * & 0 \end{pmatrix}$$

$$\sim_{C \text{ of } 1\text{-jets}} \begin{pmatrix} 0 & * & * \\ 0 & * & * \end{pmatrix}$$

which is by lemma 4.1 not a 1-jet of the matrix of a simple fat point.

In the other case, we have to consider 2 subcases, namely

$$j_1M \sim_C \begin{pmatrix} x & \alpha y & 0 & 0 \\ y & * & * & y \\ 0 & * & * & * \end{pmatrix}$$

$$\sim_C \begin{pmatrix} x & \alpha y & 0 & 0 \\ 0 & * & * & y \\ 0 & * & * & * \end{pmatrix}$$

(which is again in the first case from the beginning of this proof) or

$$j_1M \sim_C \begin{pmatrix} x & \alpha y & 0 & 0 \\ y & * & 0 & x \\ 0 & * & * & * \end{pmatrix}$$

In this last subcase, we see from the third column that either the singularity defined by M cannot be simple by lemma 4.1 (if the third column is zero), or j_1M is again in the first case of the proof by exchanging appropriate columns and rows.

□

Therefore we see that a simple fat point singularity in $(\mathbb{C}^2, 0)$ which is not a complete intersection has to be described by a matrix with 1-jet

$$\begin{pmatrix} x & y & 0 \\ 0 & \beta x & y \end{pmatrix}$$

where $\beta \in \{0, 1\}$. By the same arguments as in the proof of lemma 4.2, we see that the matrix will then be contact-equivalent to

$$\begin{pmatrix} x & y & 0 \\ 0 & x^k & y \end{pmatrix}.$$

Lemma 4.4 *The only fat point singularities in $(\mathbb{C}^2, 0)$ which are simple but not complete intersections are listed in the following table:*

Type	Equations	τ	
Ξ_k	$\begin{pmatrix} x & y & 0 \\ 0 & x^k & y \end{pmatrix}$	$k+3$	$k \geq 1$

Proof: Using T_X^1 , as in the last chapter, we can determine the versal family and calculate the possible deformations of M :

$$M' \sim_C \begin{pmatrix} x & y + \beta & \gamma \\ \alpha & x^k + \delta_{k-1}x^{k-1} + \dots + \delta_0 & y \end{pmatrix}$$

with $\alpha, \beta, \gamma, \delta_{k-1}, \dots, \delta_0 \in \mathbb{C}$.

$$\begin{pmatrix} x & y + \beta & \gamma \\ \alpha & x^k + \delta_{k-1}x^{k-1} + \dots + \delta_0 & y \end{pmatrix} \sim_C \begin{cases} \langle x^{k+2}, y \rangle & \text{for } \alpha \neq 0 & A_{k+1} \\ \langle xy, x^k + y^2 \rangle & \text{for } \alpha = 0, \gamma \neq 0 & F_{k+1}^{k,2} \\ \langle x, y^2 \rangle & \text{for } \alpha = \gamma = 0, \delta_0 \neq 0 & A_1 \\ \langle x^{k+1}, y \rangle & \text{for } \alpha = \gamma = \delta_0 = 0, \beta \neq 0 & A_k \\ \begin{pmatrix} x & y & 0 \\ 0 & x^i & y \end{pmatrix} & \text{for } \alpha = \gamma = \delta_0 = \gamma = 0 & \Xi_i, i \leq k \end{cases}$$

Since all (finitely many types of) singularities appearing here are simple, we have proved that Ξ_k is simple.

□

5 Adjacencies

When proving a classification of simple isolated singularities, a large number of adjacencies is usually determined explicitly as a byproduct or follows from these by transitivity (see [Arn], [Giu], [FK1]). In particular, knowing the complete list of adjacencies for a given series (resp. at least knowing into which series it can deform into) is vital for proving that all singularities in this series are simple.

On the other hand, when proving that a singularity, which is not part of a series, is indeed simple, it is often sufficient to know by semicontinuity of certain invariants (e.g. δ , μ) that adjacencies to non-simple singularities cannot occur.

Due to the large number of simple singularities in 4 and more variables, we only concentrate on 2 and 3 variables for computing adjacencies where the case of 3 variables may be regarded as a model for the cases with more than 3 variables. In order to determine the complete list of adjacencies for isolated simple fat point and space curve singularities, we will now start by summarizing what is known:

While the list of adjacencies from Arnold's classification [Arn] can easily be checked to be complete, there are known gaps in Giusti's list, as was first observed when additional adjacencies were found for space curves singularities by Goryunov [Gor]. These gaps were closed in [S-V] for the adjacencies among the isolated simple complete intersection space curve singularities. The case of adjacencies to plane curve singularities has later been treated by computer algebra methods in [FK2].

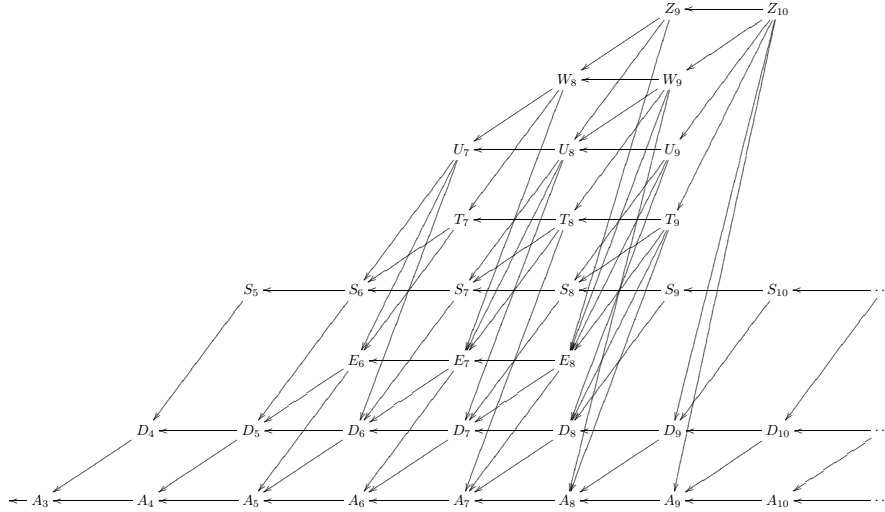


Diagram 1: Adjacencies of simple ICIS space curve singularities due to [Arn],[Giu],[Gor],[S-V],[FK2]

In the case of simple fat point singularities, the adjacencies of the series

Ξ_k already follow immediately from the proof of lemma 4.4. So the remaining considerations in this case concern Giusti's list. Here it turns out that the adjacency table given in [Giu] is not just incomplete, but even contains non-existent adjacencies. Therefore we state corrected versions of the adjacency tables for the series F , H and I here⁶:

Adjacencies for singularities of type $F_{n+p-1}^{n,p}$:

$$\langle \alpha x + \beta y + xy, a_1 x + \dots + x^n + b_1 y + \dots + y^p \rangle,$$

$$\sim_C \left\{ \begin{array}{lll} \langle x, y^p \rangle & \text{for } \alpha \neq 0 & A_{p-1} \\ \langle y, x^n \rangle & \text{for } \alpha = 0, \beta \neq 0 & A_{n-1} \\ \langle y^{p+1}, x \rangle & \text{for } \alpha = \beta = 0, a_1 \neq 0 & A_p \\ \langle x^{n+1}, y \rangle & \text{for } \alpha = \beta = a_1 = 0, b_1 \neq 0 & A_n \\ \langle xy, x^i + y^j \rangle & \text{for } \alpha = \beta = a_1 = b_1 = 0 & F_{i+j-1}^{i,j}, i \leq n, j \leq p \end{array} \right.$$

Adjacencies for singularities of type H_{n+3} :

$$\langle x^2 + \alpha x + a_1 y + \dots + a_{n-1} y^{n-1} + y^n, xy^2 + \beta x + \gamma y + bxy + cy^2 + dy^3 \rangle$$

$$\sim_C \left\{ \begin{array}{lll} \langle x, y^{n+2} \rangle & \text{for } \alpha \neq 0 & A_{n+1} \\ \langle y, x^5 \rangle & \text{for } \alpha = 0, a_1 \neq 0 & A_4 \\ \langle y^n, x \rangle & \text{for } \alpha = a_1 = 0, \beta \neq 0 & A_{n-1} \\ \langle x^2, y \rangle & \text{for } \alpha = a_1 = \beta = 0, \gamma \neq 0 & A_1 \\ \langle x^2 + y^2, xy \rangle & \text{for } b \neq 0, a_2 \neq c^2 & F_3^{2,2} \\ \langle x^2 + y^3, xy \rangle & \text{for } b \neq 0, a_2 = c^2, a_3 \neq cd & F_4^{2,3} \\ \langle x^2 + y^4, xy \rangle & \text{for } b \neq 0, a_2 = c^2, a_3 = cd, a_4 \neq d^2 & F_5^{2,4} \\ \langle x^2 + y^i, xy \rangle & \text{for } b \neq 0, a_2 = c^2, a_3 = cd, a_4 = d^2, \\ & a_5 = \dots = a_{i-1} = 0, a_i \neq 0, i \leq n & F_{i+1}^{2,i} \\ \langle x^2 + y^2, xy \rangle & \text{for } b = 0, a_2 \neq 0, c \neq 0 & F_3^{2,2} \\ \langle x^3 + y^3, xy \rangle & \text{for } b = 0, a_2 \neq 0, c = 0 & F_5^{3,3} \\ \langle x^2 + y^2, xy \rangle & \text{for } b = a_2 = 0, c \neq 0 & F_3^{2,2} \\ \langle x^2, y^3 \rangle & \text{for } b = a_2 = c = 0, d \neq 0 & G_5 \\ \langle x^2 + y^{n-1}, xy^2 \rangle & \text{for } b = a_2 = \dots = a_{n-2} = c = d = 0, \\ & a_{n-1} \neq 0 & H_{n+2} \end{array} \right.$$

⁶The adjacency lists for the singularities of types G_5 and G_7 did not contain any mistakes.

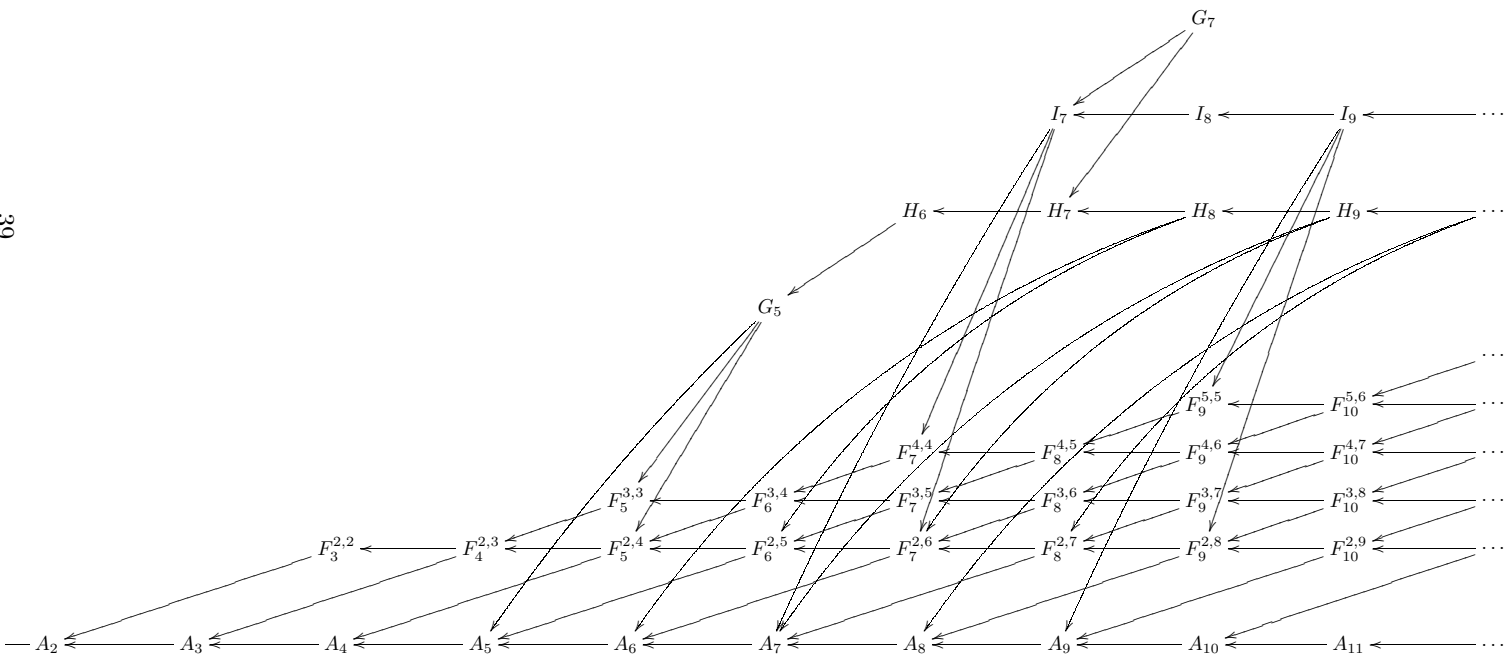


Diagram 4: Adjacencies of simple ICIS fat point singularities from Giusi's classification [Giu] corrected according to the adjacency lists

Adjacencies for singularities of type I_{2p-1} :

$$\begin{aligned}
& \langle x^2 + y^3 + ay^2 + \alpha x + \beta y, \\
& \quad \gamma x + \delta y + b_1 xy + \dots + b_{p-2} xy^{p-2} + c_2 y^2 + \dots + c_{p-1} y^{p-1} + y^p \rangle \\
& \sim_C \left\{ \begin{array}{lll} \langle x, y^{p+1} \rangle & \text{for } \alpha \neq 0 & A_p \\ \langle y, x^{2p} \rangle & \text{for } \alpha = 0, \beta \neq 0 & A_{2p-1} \\ \langle y^{2p}, x \rangle & \text{for } \alpha = \beta = 0, \gamma \neq 0 & A_{2p-1} \\ \langle x^2, y \rangle & \text{for } \alpha = \beta = \gamma = 0, \delta \neq 0 & A_1 \\ \langle x^2 + y^i, xy \rangle & \text{for } b_1 \neq 0, i \leq 2p-2 & F_{i+1}^{2,i} \\ \langle x^i + y^j, xy \rangle & \text{for } b_1 = 0, a \neq 0, i, j \leq p & F_{i+j-1}^{i,j} \\ \langle x^2 + y^2, xy \rangle & \text{for } b_1 = a = 0, c_2 \neq 0 & F_3^{2,2} \\ \langle x^2, y^3 \rangle & \text{for } b_1 = a = c_2 = 0, c_3 \neq 0 & G_5 \\ \langle x^2 + y^3, xy^2 \rangle & \text{for } b_1 = a = c_2 = c_3 = 0, b_2 \neq 0 & H_6 \\ \langle x^2 + y^3, xy^{p-2} \rangle & \text{otherwise} & I_{2p-2} \end{array} \right.
\end{aligned}$$

Adjacencies for singularities of type I_{2q+2} :

$$\begin{aligned}
& \langle x^2 + ay^2 + y^3 + \alpha x + \beta y, \\
& \quad \gamma x + \delta y + b_1 xy + \dots + b_{q-1} xy^{q-1} + xy^q + c_2 y^2 + \dots + c_{q+1} y^{q+1} \rangle \\
& \sim_C \left\{ \begin{array}{lll} \langle x, y^{q+3} \rangle & \text{for } \alpha \neq 0 & A_{q+2} \\ \langle y, x^{2q+2} \rangle & \text{for } \alpha = 0, \beta \neq 0 & A_{2q+1} \\ \langle y^{2q+2}, x \rangle & \text{for } \alpha = \beta = 0, \gamma \neq 0 & A_{2q+1} \\ \langle x^2, y \rangle & \text{for } \alpha = \beta = \gamma = 0, \delta \neq 0 & A_1 \\ \langle x^2 + y^i, xy \rangle & \text{for } b_1 \neq 0, i \leq 2q & F_{i+1}^{2,i} \\ \langle x^i + y^j, xy \rangle & \text{for } b_1 = 0, a \neq 0, i, j \leq q+1 & F_{i+j-1}^{i,j} \\ \langle x^2 + y^2, xy \rangle & \text{for } b_1 = a = 0, c_2 \neq 0 & F_3^{2,2} \\ \langle x^2, y^3 \rangle & \text{for } b_1 = a = c_2 = 0, c_3 \neq 0 & G_5 \\ \langle x^2 + y^3, xy^2 \rangle & \text{for } b_1 = a = c_2 = c_3 = 0, b_2 \neq 0 & H_6 \\ \langle x^2 + y^3, y^{q+1} \rangle & \text{otherwise} & I_{2q+1} \end{array} \right.
\end{aligned}$$

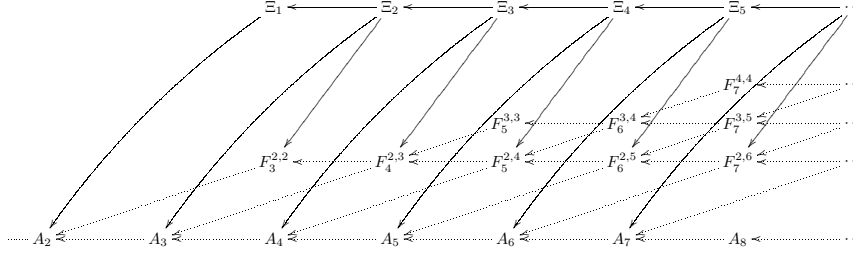


Diagram 5: Adjacencies of simple non-complete-intersection fat point singularities. As singularities of type Ξ_k cannot deform into singularities of types G , H and I , only singularity types F and A appear in this diagram, adjacencies among those are drawn as dotted lines.

The last item that remains to be determined is the complete list of adjacencies for simple isolated space curve singularities which are not complete intersections⁷. Due to the large number of simple space curve singularities, this turns out to be a rather lengthy calculation which should begin at the singularities with the lowest values for the invariants τ and δ and always involves the same steps for each given simple singularity. Therefore we only sketch the basic concept of this calculation, the detailed results for each singularity and each step can be found at

<http://www.mathematik.uni-kl.de/~anne/adjCMcod2.html>

Step 1: Determining candidates for new adjacencies

For a given singularity, we first consider all singularities, whose Tjurina number is smaller than the one of the given singularity, as possible target of an adjacency. From this list, we exclude, of course, those singularities for which it has been shown in the proof of simplicity that the adjacency does not exist and mark the ones which have been shown to exist.

Using the semicontinuity of the invariants δ and μ and the fact that δ has to be constant in a μ -constant family (cf. [B-G]), we can then exclude some more singularities from the list. In the case of the singularity S_6^* , for example, the invariants are $\tau = 7$, $\mu = 6$ and $\delta = 4$ and thus adjacencies to the singularities A_6 ($\mu = 6$, $\delta = 3$) and E_6 ($\mu = 6$, $\delta = 3$) are excluded by the latter condition.

Step 2: Finding adjacencies for which τ drops exactly by one

For finding these adjacencies, we have to study the structure of the base space of the versal family of the given singularity more closely. To this end, let t_1, \dots, t_τ denote the parameters of the versal family and consider

⁷The classification of these singularities can be found in [FK1]

the relative T^1 of the family as a $\mathbb{C}[t_1, \dots, t_\tau]$ -module. The flattening stratification⁸ of this module determines the strata in the base space of the versal family on which the Tjurina number is constant.

Since all simple singularities from our list are quasihomogeneous, we know that we also have an Euler relation for the given singularity and thus the $(\tau-1)$ -th Fitting ideal of the relative T^1 is the maximal ideal at the origin. The stratum on which τ drops exactly by one is determined by $\text{Fitt}_{(\tau-2)}(T_{rel}^1)$ ideal on the complement of $\text{Fitt}_{(\tau-1)}(T_{rel}^1)$.

On each primary component of $\text{Fitt}_{(\tau-2)}(T_{rel}^1)$, we have exactly one type of singularity. So, it now suffices to determine a primary decomposition of this ideal and determine the type of singularity on each component. The types of singularities appearing there are exactly those of the correct Tjurina number τ to which an adjacency exists. This allows us to exclude resp. to mark further singularities in our list of candidates (of course also using the transitivity of adjacencies again).

During the computations for all simple space curve singularities, it turned out that after this step no unmarked candidates were left in all cases except for the singularities $E_k \vee L$, $k \in \{6, 7, 8\}$.

Step 3: Treating the remaining cases

For the singularities $E_k \vee L$, $k \in \{6, 7, 8\}$, there is only one remaining candidate for an adjacency, namely $E_k \vee L \longrightarrow A_k$. To exclude this adjacency, we first observe that a versal family of an $E_k \vee L$ singularity is of the form

$$\begin{pmatrix} z & \alpha & \beta \\ t_1 & x & y \end{pmatrix},$$

where α and β are suitable polynomials in x, y and the parameters t_2, \dots, t_τ . Moreover, it is easy to see that a plane curve singularity which is not an A_1 can only occur if $t_1 \neq 0$. But for $t_1 \neq 0$ and $t_2 = \dots = t_\tau = 0$, we obtain a U_{k+1} singularity. Moreover, any singularity with $t_1 \neq 0$ in the versal family of $E_k \vee L$ also appears in the versal family of U_{k+1} as one can check by direct computation. Now we already know from [FK2] that U_{k+1} cannot deform into A_k . This excludes the last remaining candidates.

⁸From the computational point of view, determining the flattening stratification means computing a presentation matrix of the module, which is described in [FK2], and then forming the Fitting ideals. Since computing minors of matrices with polynomial entries is quite expensive, we first only consider the stratum on which τ drops by exactly one which is given by the vanishing of the 2-minors and non-vanishing of the 1-minors of the presentation matrix. It turns out during the computation that we can actually avoid studying the stratum on which τ drops by 2 in all cases.

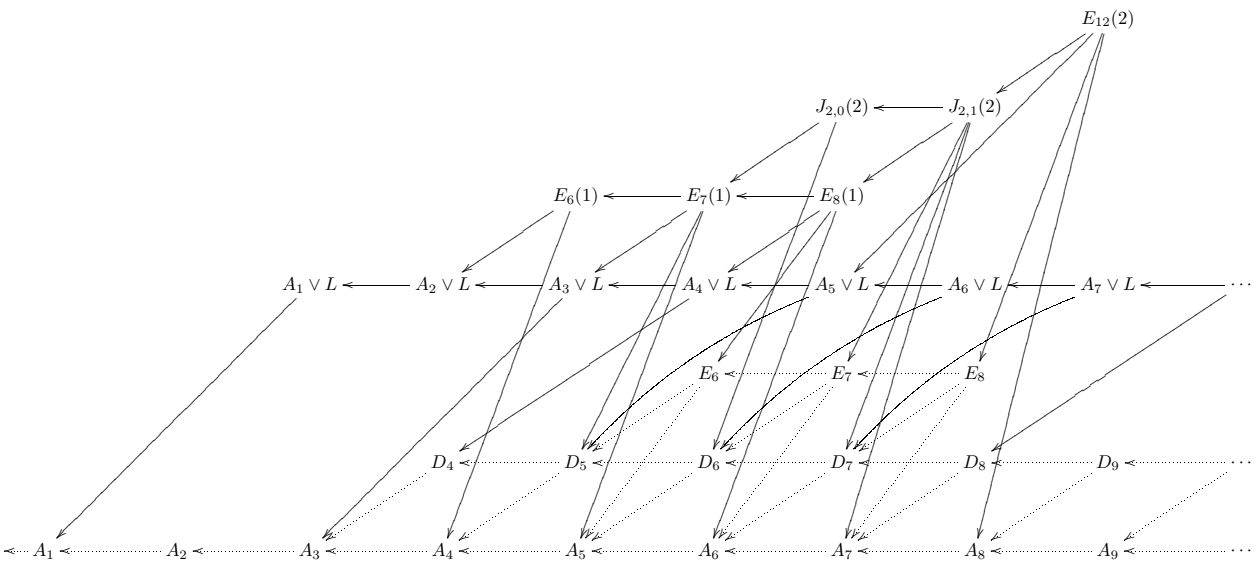


Diagram 6: Adjacencies of simple space curve singularities of multiplicity 3
(Adjacencies of plane curves from Arnold's list are shown as dotted lines.)

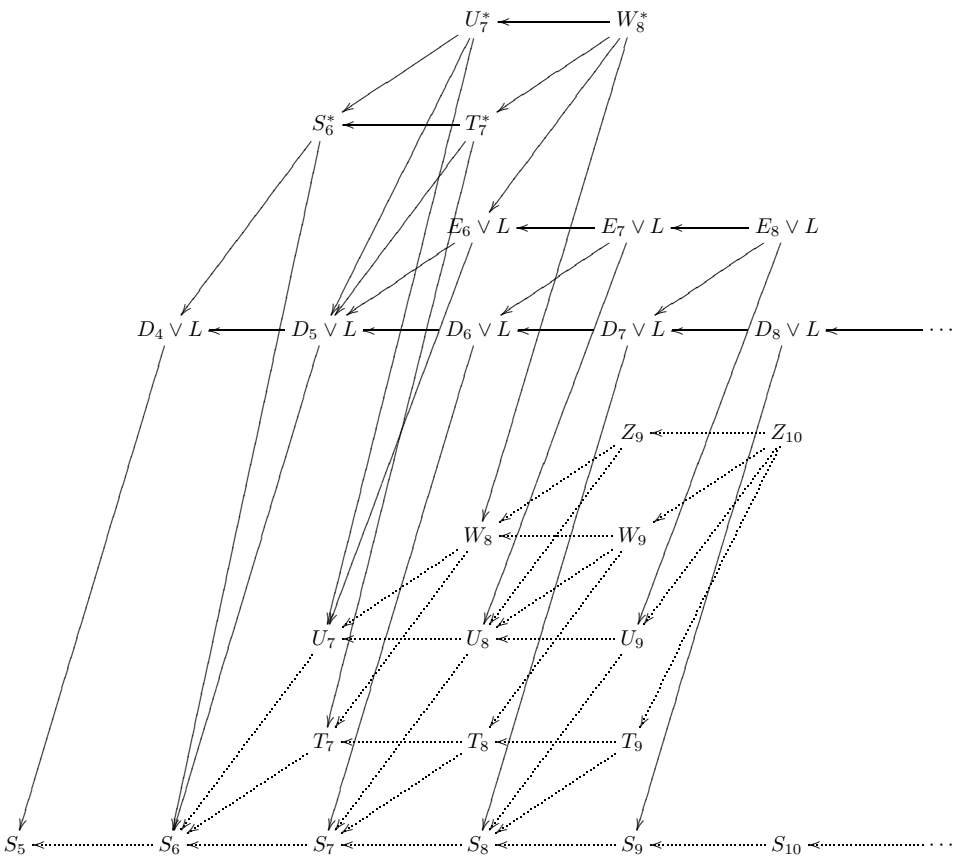


Diagram 7: Adjacencies among simple space curve singularities of multiplicity 4 to space curve singularities of multiplicity 4 (Adjacencies from [Giu], [S-V], [FK2] are shown as dotted lines.)

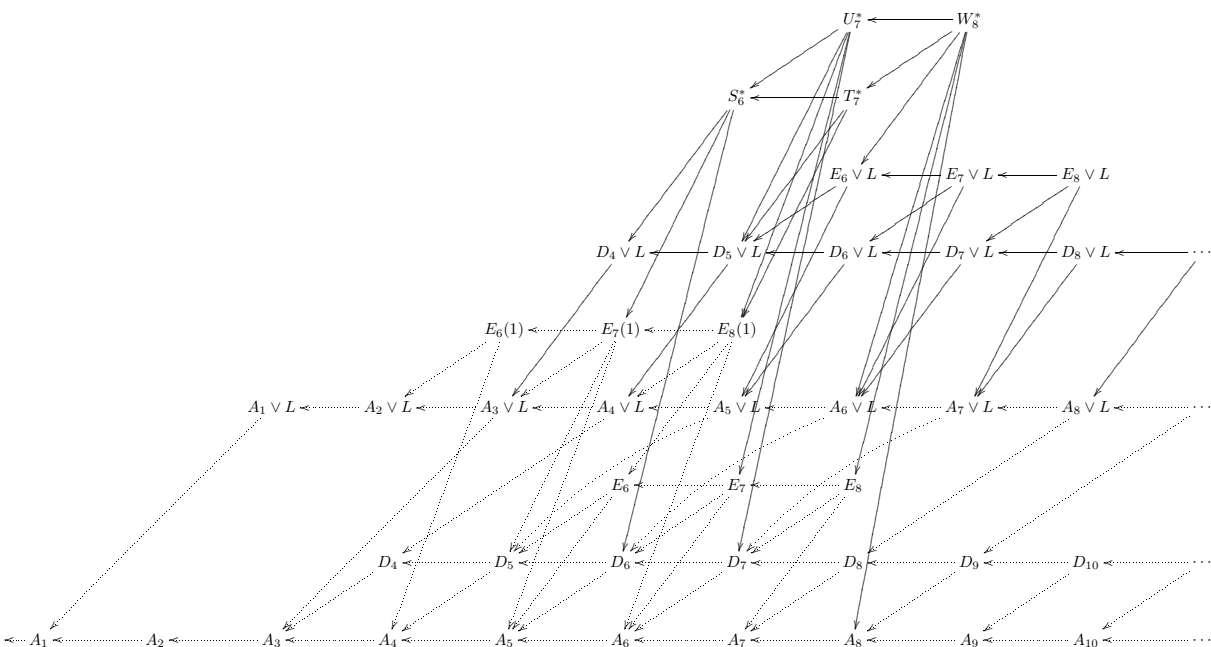


Diagram 8: Adjacencies of simple space curve singularities of multiplicity 4 into singularities of multiplicity at most 3 (Adjacencies from [Arn], [FK2] are shown as dotted lines. Note that further adjacencies to singularities of multiplicity 3 do exist as combination of an adjacency of the previous diagram and diagram 1 due to transitivity)

6 Appendix

For readers' convenience, Arnold's list of simple hypersurface singularities from [Arn] and the complete lists of previously known simple Cohen-Macaulay codimension 2 singularities (cf. [Giu] and [FK1]) are listed in this Appendix.

Type	Presentation Matrix	μ	τ	δ	
A_k	$x^{k+1} + y^2$	k	k	$\lfloor \frac{k}{2} \rfloor$	$k \geq 1$
D_k	$x^2y + y^{k-1}$	k	k	$\lfloor \frac{k+2}{2} \rfloor$	$k \geq 4$
E_6	$x^3 + y^4$	6	6	3	
E_7	$x^3 + xy^2$	7	7	4	
E_8	$x^3 + y^5$	8	8	4	

Table 1: The simple hypersurface singularities

Type	Presentation Matrix	μ	τ	
A_k	$\langle y, x^{k+1} \rangle$	k	k	$k \geq 1$
$F_{q+r-1}^{q,r}$	$\langle xy, x^q + y^r \rangle$	$q + r - 1$	$q + r$	$q, r \geq 2$
G_5	$\langle x^2, y^3 \rangle$	5	7	
G_7	$\langle x^2, y^4 \rangle$	7	10	
H_{q+3}	$\langle x^2 + y^q, xy^2 \rangle$	$q + 3$	$q + 5$	$q \geq 3$
I_{2q-1}	$\langle x^2 + y^3, y^q \rangle$	$2q - 1$	$2q + 1$	$q \geq 4$
I_{2r+2}	$\langle x^2 + y^3, xy^r \rangle$	$2r + 2$	$2r + 4$	$r \geq 3$
Ξ_k	$\begin{pmatrix} x & y & 0 \\ 0 & x^k & y \end{pmatrix}$		$k + 3$	$k \geq 1$

Table 2: The simple fat point singularities in $(\mathbb{C}^2, 0)$

Type	Presentation Matrix	μ	τ	δ	
S_{n+3}	$(x^2 + y^2 + z^n, yz)$	$n + 3$	$n + 3$	$\lfloor \frac{n+6}{2} \rfloor$	$n \geq 2$
T_7	$(x^2 + y^3 + z^3, yz)$	7	7	4	
T_8	$(x^2 + y^3 + z^4, yz)$	8	8	5	
T_9	$(x^2 + y^3 + z^5, yz)$	9	9	5	
U_7	$(x^2 + yz, xy + z^3)$	7	7	4	
U_8	$(x^2 + yz + z^3, xy)$	8	8	5	
U_9	$(x^2 + yz, xy + z^4)$	9	9	5	
W_8	$(x^2 + z^3, y^2 + xz)$	8	8	4	
W_9	$(x^2 + yz^2, y^2 + xz)$	9	9	5	
Z_9	$(x^2 + z^3, y^2 + z^3)$	9	9	5	
Z_{10}	$(x^2 + yz^2, y^2 + z^3)$	10	10	5	

Table 3: Simple space curve singularities in $(\mathbb{C}^3, 0)$, Part 1: ICIS

Type	Presentation Matrix	μ	τ	δ	
$A_{k-3} \vee L$ $k \geq 4$	$\begin{pmatrix} z & y & x^{k-3} \\ 0 & x & y \end{pmatrix}$	$k-2$	$k-1$	$\frac{k}{2}$ $\frac{k-1}{2}$	k even k odd
$E_6(1)$	$\begin{pmatrix} z & y & x^2 \\ x & z & y \end{pmatrix}$	4	5	2	
$E_7(1)$	$\begin{pmatrix} z+x^2 & y & x \\ 0 & z & y \end{pmatrix}$	5	6	3	
$E_8(1)$	$\begin{pmatrix} z & y & x^3 \\ x & z & y \end{pmatrix}$	6	7	3	
$J_{2,k}(2)$ $k \in \{0, 1\}$	$\begin{pmatrix} z+x^2 & y & x^{k+2} \\ 0 & z & y \end{pmatrix}$	6 7	7 8	4	$k=0$ $k=1$
$E_{12}(2)$	$\begin{pmatrix} z & y & x^3 \\ x^2 & z & y \end{pmatrix}$	8	9	4	
$D_{k+4} \vee L$ $k \geq 0$	$\begin{pmatrix} z & 0 & x^{k+2}-y^2 \\ 0 & x & y \end{pmatrix}$	$k+5$	$k+6$	$\frac{k+8}{2}$ $\frac{k+7}{2}$	k even k odd
$E_6 \vee L$	$\begin{pmatrix} z & -y^2 & -x^3 \\ 0 & x & y \end{pmatrix}$	7	8	4	
$E_7 \vee L$	$\begin{pmatrix} z & x^3-y^2 & 0 \\ 0 & x & y \end{pmatrix}$	8	9	5	
$E_8 \vee L$	$\begin{pmatrix} z & -y^2 & -x^4 \\ 0 & x & y \end{pmatrix}$	9	10	5	
S_6^*	$\begin{pmatrix} z & x & y \\ 0 & y & x^2-z^2 \end{pmatrix}$	6	7	4	
T_7^*	$\begin{pmatrix} z & x & y \\ 0 & y & x^2-z^3 \end{pmatrix}$	7	8	4	
U_7^*	$\begin{pmatrix} z & xy & x^2 \\ x & z & y \end{pmatrix}$	7	8	4	
W_8^*	$\begin{pmatrix} z & y^2 & x^2 \\ x & z & y \end{pmatrix}$	8	9	4	

Table 4: Simple space curve singularities in $(\mathbb{C}^3, 0)$, Part 2:
Non-complete-intersections

References

- [AGV] Arnold,V., Gusein-Zade,S., Varchenko,A.: *Singularities of Differentiable Maps I*, Birkhäuser (1985)
- [Arn] Arnold,V.: *Normal Forms for Functions near degenerate Critical Points...*, FAA **6**, 254–272, (1972)
- [Art] Artin,M.: *Lectures on Deformations of Singularities*, Tata Institute of Fundamental Research, Bombay (1976)
- [B-G] Buchweitz,R., Greuel,G.: *The Milnor Number and Deformations of Complex Curve Singularities*, Invent.Math. **58**, 241–281 (1980)
- [Bur] Burch,L.: *On ideals of finite homological dimension in local rings.*, Proc. Camb. Phil. Soc. **64**, 941–948 (1964)
- [FK1] Frühbis-Krüger,A.: *Classification of Simple Space Curve Singularities*, Comm.in Alg. **27(8)**, 3993–4013 (1999)
- [FK2] Frühbis-Krüger,A.: *Partial Standard Bases for Families*, in Proceedings of ICMS 2002, World Scientific Publishing, 228–238 (2002)
- [FK3] Frühbis-Krüger,A.: *Partial Standard Bases as a Tool for Studying Families of Singularities*, to appear in J. Symb.Comp.
- [Gib] Gibson,C., Hobbs,C.: *Simple Singularities of Space Curves*, Math.Proc.Camb.Philos.Soc. **113**, 297–310 (1993)
- [Giu] Giusti,M.: *Classification des singularités isolées simples d’intersections complètes*, Proc.Symp.Pure Math. **40**, 457–494 (1983)
- [Gor] Goryunov,V.: *Singularities of projections of full intersections*, Journal of Soviet Math. **27(3)**, 2785–2811 (1984)
- [K-S] Kolgushkin,P., Sadykov,R.: *Simple singularities of multigerms of curves*, Rev. Mat. Compl. **XIV (2)**, 311–344 (2001)
- [Sch] Schaps,M.: *Deformations of CohenMacaulay schemes of codimension 2 and nonsingular deformations of space curves*, Amer. J. Math. **99**, 669684 (1977)
- [Sin] Greuel,G., Pfister,G., Schönemann,H.: *Singular version 2.2 User Manual*, to appear in Report on Computer Algebra, Centre for Computer Algebra, University of Kaiserslautern (2003)
- [S-V] Schröter,C., Schuba,S.: *The deformations of the simple space curve singularities*, Math. Z. **229**, 607–614 (1998)
- [Tju] Tjurina,G.N.: *Absolute isolatedness of rational singularities and triple rational points*, Func.Anal.Appl. **2**, 324–333 (1968)

- [Wal] Wall, C.T.C.: *Classification of unimodal isolated singularities of complete intersections*, in Proc. Symp. in Pure Math. **40ii** (Singularities) (ed. P. Orlik) Amer. Math. Soc., 625–640 (1983)